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CALCULATIONS
IN
HYDRAULIC ENGINEERING

A PRACTICAL TEXT-BOOK FOR THE USE OF STUDENTS,
DRAUGHTSMEN, AND ENGINEERS

WITH NUMEROUS ILLUSTRATIONS AND EXAMPLES

BY

T. CLAXTON FIDLER, M.INST.C.E.
PROFESSOR OF ENGINEERING, UNIVERSITY COLLEGE, DUNDEE

PART I.

FLUID PRESSURE, AND THE CALCULATION OF ITS EFFECTS IN ENGINEERING STRUCTURES

LONGMANS, GREEN, AND CO.
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PREFACE

THIS little book does not profess to be a treatise on hydraulics, but relates to those calculations which have to be made—many of them very frequently, and others perhaps more rarely, in connection with works of hydraulic engineering.

It is hoped that the complete book may be found useful to those practical engineers, assistants, and draughtsmen, who are often engaged in carrying out such calculations, as well as to engineering students; and with this object in view, it has been written in the language of practical men rather than in that of the schools or of the mathematician. In all cases, it has been the author's desire to discuss the rational groundwork of these problems in the simplest and plainest terms; and the student who happens to be accustomed to a more rapid mathematical treatment may perhaps find these preliminary discussions unnecessary; but they may nevertheless be acceptable to other students, whose training has been of a more practical kind, and whose ideas have been cast in a somewhat different mould.

Some apology should perhaps be offered for the way in which diagrams B and C have been drawn in Fig. 24. If the diagrams had been inverted they would have better indicated the negative character of the quantities; but the chief thing to be illustrated was the similarity of form between the curves A, C, and E; and it is not likely that any error could arise as to the character of the bending moment.

In dealing with the effects of fluid pressure, a good many of the calculations that have to be made are too simple to need any extended illustration in this book; and for that reason, a great part of its contents is devoted to those problems which are not so simple, or to those which seem to have received but little consideration at the hands of other writers.

Among these matters may be mentioned the forces which result from the axial fluid stress in a curved or in a straight pipe. To the mathematician this question will no doubt appear to be a simple one; but the practical man will certainly call to mind the innumerable cases in which these forces are unattended with any perceptible results; and it was therefore necessary to make some attempt to set in categorical order the several conditions under which these forces really come into action, either upon the pipe or upon surrounding bodies. The question is discussed and illustrated from various points of view in Chapters III., IV., and V., and the conclusions in regard to the strength and stability of pipe-lines are only reached in Chapter VI. The author has met with a large number of instances in which these theorems have an important application in practice.

The stability of floating bodies is treated with especial reference to engineering structures, such as floating docks, pontoons, and the caissons that are sometimes used for bridge foundations. The reader will have no difficulty in applying the method used for these examples to the case of cellular caissons of other shapes.

The bending stresses and the elastic deflection of loaded pontoons are considered in the next ensuing chapter.

Some experimental measurements of the displacing forces, in the fluid arch and in the fluid column, are described in detail in the Appendix, which also contains a short series of experiments upon the direct tensile strength of various forms of socket joints.

In regard to those calculations of flow and discharge which must be included in the second part of the book, the author would desire to institute, if it should be possible, a few supplementary experiments, before completing this part for publication.

T. C. F.

UNIVERSITY COLLEGE, DUNDEE,
April, 1898.

CONTENTS

PAGE
INTRODUCTION I

PART I.

FLUID PRESSURE, AND THE CALCULATION OF IT'S EFFECTS IN ENGINEERING STRUCTURES.

CHAPTER I.

PRACTICAL CONCEPTIONS OF ELEMENTARY PRINCIPLES.

ART.
1. Fluid Pressure 5
2. Fluid Pressure of Uniform Intensity . . 6
3. Hydrostatic Pressure of Varying Intensity 7

CHAPTER II.

THE ACTION OF FLUID PRESSURE OF UNIFORM INTENSITY.

4. Centre of Action—Its position in plane surfaces of regular figure 10
5. The Pressure directly opposed by Contrary Forces or Resistances . . 14
6. Resolution of the Fluid Pressure into Components whose Direction is known or mechanically determined — Illustration by practical examples 14

CHAPTER III.

CYLINDERS, PIPES, AND BENDS UNDER UNIFORM PRESSURE.

7. Tangential Stress or Hoop-tension—Mean stress and maximum stress in thin and in thick cylinders . . . 19
8. Axial Fluid Stress—Stresses in a cubical vessel and in a closed cylinder —Axial stress in a line of water-main 21
9. Displacing Forces in a Pipe-bend of Polygonal Form—Axial stress equilibrated by forces or resistances applied at the sides or at the angles of the polygon 23
10. Displacing Forces in a Circular or Segmental Bend—Axial stress equilibrated by a uniformly distributed pressure normal to the curve . 28
11. Axial Stress in the Fluid and in the Pipe—Example of the summation of similar axial stresses 30

ART. PAGE
12. The Displacing Force resisted by Longitudinal Stress opposite to the axial fluid stress—Observations on the different cases that may arise in practice 32
13. Expansion Joints—The longitudinal stress in the pipe annulled, or mechanically determined in magnitude and direction 33
14. The Pipe-bend acting as a Girder—Construction of the diagram of moments for a segmental bend with sliding ends 35
15. The Pipe-bend acting as a Cantilever—The steam-pipes of marine engines—Example of the calculable stresses 37
16. Guard Bolts—Their function and the limitations of their action . 39
17. The Transverse Strength of Cylindrical Pipes—Calculation of the maximum fibre stress due to known bending moments 40
18. The Transverse Strength of Pipe Joints—Calculation of maximum stress in flange-bolts and in riveted joints . . 41

CHAPTER IV.

THE BUCKLING TENDENCY IN STRAIGHT PIPES UNDER UNIFORM PRESSURE.

19. Axial Stress in the Straight Fluid Column—Buckling action of the fluid stress and its proximate cause 44
20. Displacing Forces due to Elastic Deflection—Their development during the buckling of a straight pipe 45
21. The Conditions of Equilibrium broadly considered—Approximate calculation of the critical fluid pressure 46
22. Exact Determination of the Critical or Buckling Pressure—Analogy with the case of a slender elastic column under direct load 48
23. Effective Length for Flexure and Effective Axial Thrust—The buckling tendency annulled in some cases by longitudinal stress in the pipe—Its action in a pipe subjected to longitudinal tension . . 51

CHAPTER V.

FLUID ARCHES.

24. Pressure Pipes used as Tubular Arches—General effect of the displacing force in such cases 54
25. Direct Longitudinal Stress in charged Tubular Arches—Examples of the calculation of compressive and of tensile stresses under given loads 55
26. Equilibrated Fluid Arches—The theoretical form and construction of tubular arches equilibrated under given loads by fluid stress alone . 59
27. Summary of Results—List of the several forces or stresses concerned in the equilibrium of a curved pipe 62

CHAPTER VI.

THE STABILITY OF A LINE OF PIPE.

28. Continuous Lines of Flanged or Riveted Pipe—The displacing forces annulled by longitudinal tension in the pipe—Requisite strength of the joints 63
29. Pipes Fitted with Expansion Joints—The longitudinal tension annulled—The displacing force to be otherwise resisted 64

ART. PAGE

30. Lines of Socket-jointed Pipe—The combination of two or more resistances—The tensile strength of socket joints—The earth-load upon vertical bends in a buried pipe 64

30A. Joints Turned and Bored—The limited tensile strength of such joints . 66

CHAPTER VII.

HYDROSTATIC PRESSURE OF VARYING INTENSITY.

31. The Distribution of Hydrostatic Pressure—Illustration by diagram and model 67

32. The Measurement of Hydrostatic Pressure—The mean pressure and the total pressure upon plane surfaces of regular figure 68

33. The Centre of Action—Its position upon the same plane surfaces when the water-line coincides with their upper edge 71

34. Vertical Walls of Irregular Figure—Example of the calculation for the case of a dock caisson 72

35. Surfaces submerged in Deep Water—General method of treatment . 75

36. Contrary Horizontal Pressures—Effective pressure on a lock gate . 76

37. Hydrostatic Pressure upon Inclined and upon Curved Surfaces—The horizontal and vertical components—Vertical buoyant pressure upon an overhanging surface . 77

CHAPTER VIII.

FLOATING VESSELS.

38. Equilibrium of Floating Bodies in General—Special application to engineering structures 81

39. Buoyancy—The "displacement" of vessels . . 81

40. Equilibrium of the Vertical Forces 82

41. Vertical Motion under the Vertical Forces—Examples of stable, indifferent, and unstable equilibrium as regards the vertical motion . 82

42. Lines of Action of the Vertical Forces—The section of the vessel considered as a diagram of buoyant pressures—Centre of gravity and centre of buoyancy 84

43. Conditions of Equilibrium in Still Water—Righting moments and upsetting moments 85

44. Stability of a Closed Vessel wholly Submerged, and of a submerged homogeneous solid 86

45. Stability of a Floating Gas-holder, and of an Inflated Bridge Caisson 86

46. Stability of a Water-logged Vessel with Thin Sides 88

47. Stability of a Cylindrical Pontoon floating on its Side—The fixed "metacentre" 89

48. Stability of a Rectangular Pontoon in the upright position—The curve of buoyancy—Calculation of its co-ordinates and of the metacentric height—Examples 90

49. Curve of Buoyancy for Rectangular Vessels—The construction of the curve completed for all angles of roll 94

50. Diagram of Righting Moments 97

51. Cylindrical Vessel floating on End—Its stability or instability . 98

52. Annular Vessel, or Double-skinned Cylindrical Caisson—Its metacentric height 99

ART. PAGE
53. Negative Metacentric Height—Case of submerged tank partially filled, or submerged diving machine 101
54. Effect of Water-ballast in Compartments—Calculation of the reduced metacentric height 104

CHAPTER IX.

BENDING STRESSES IN FLOATING PONTOONS.

55. Distribution of the Vertical Forces—The buoyant pressure generally known but affected by elastic deflection 107
56. General Expressions for the Bending Moment and the Vertical Shearing Force 108
57. Pontoon carrying a Central Load—The stresses calculated for an inflexible and for a flexible pontoon 109
58. Pontoon carrying a Pair of Equal Loads symmetrically disposed—Calculation of the bending stress and construction of the curve of deflection . 114
59. A Series of concentrated Loads imposed at Three or More Points at known Distances 118
60. Load uniformly distributed over a Portion of the Pontoon's Length—Diagram of moments and curve of deflection 119
61. Pontoon carrying the Weight of a Long Girder—Relative deflections of girder and pontoon 122
62. Long Pontoon carrying a Short Ship or Girder . . 129
63. Greatest Bending Moment due to the opposed vertical forces . . 132
64. Greatest Bending Moment in the Ship and in the Pontoon . . 133

APPENDIX A.

EXPERIMENTAL MEASUREMENT OF DISPLACING FORCES.

1. Equilibrated Fluid Arch or Vertical Bend—Description of apparatus employed and of results observed 135
2. Experimental Buckling of the Straight Pipe—Observation of critical pressure 136
3. Buckling of the Fluid Arch 140

APPENDIX B.

FLUID CATENARY.

Determination of the co-ordinates of the curve . . 142

APPENDIX C.

EXPERIMENTS ON THE TENSILE STRENGTH OF SOCKET JOINTS.

Tabulated results 145

APPENDIX D.

CURVE OF DEFLECTION IN A PIPE BUCKLING UNDER INTERNAL PRESSURE.

The form of the curve and the value of the critical pressure 150

INDEX 153

INTRODUCTION.

We often hear it said that engineering is gradually assuming the character of an exact science; or, to speak more correctly, that the practice of the engineer is becoming little else than an application of the exact sciences; and the saying conveys one of those half-truths which are so apt to mislead people when they do not know the other half of the truth.

Of course engineering is something more than a science, and more than an application of all the sciences, for some of its weightiest and most difficult problems are unknown to any of them.

Science, for example, has nothing to say about the adaptation of means to an end—a question which presents itself at every turn, and constantly demands the engineer's most careful study in the execution as well as in the design of his works.

In the preparation of his design, it is perhaps almost indispensable that the principles of mechanical theory should be consulted; but the *first* essential and the *most* important object to be kept in view, is *not* compliance with scientific rule, but rather the achievement of a definite and clearly realized *purpose.*

And the purpose to be achieved in the design of each detail is not generally a simple or a single purpose, but is often a legion of mechanical purposes; while the conditions to be observed are not only those which theory prescribes, but include a multitude of others which are of equal importance.

The mental faculty by whose aid the engineer marshals before him, in their right order, the purposes to be achieved and the conditions to be observed, and by which he goes on to select or devise the right means for the attainment of his end, is a faculty that cannot be cultivated by the usual methods of the educationist, can scarcely be acquired by a university training (though it may perhaps be exercised), and cannot be very much strengthened by the reading of books.

However, science can tell us what it knows about law; and the engineering purpose can never be achieved *except* by a careful compliance with mechanical laws. If engineering does not consist in a knowledge of these unalterable principles, yet it would be absurd to underrate their importance. It is perfectly true, also, that in modern times the practice of engineers is, more than ever, guided by a constant reference to theoretical calculations; and in no branch of construction is this to be seen more conspicuously than in works of hydraulic engineering, for here calculations are constantly being made to determine a number of questions on which must depend the broad outlines, the minute details, and almost every feature of the work.

The writings that exist on this subject show us, indeed, that it forms one of the oldest parts of engineering theory; and from the time of Leonardo da Vinci men have been striving to reduce to order the principles that govern the flow of water, while the aqueducts of old Rome seem to prove that this people must have known a good deal more about the matter than we sometimes give them credit for.

Such knowledge as they possessed must certainly have been derived mainly from experimental observations, and there is evidence enough to show that its extent was somewhat limited; but we all know that since their time the exact sciences have made enormous progress. And yet it is a remarkable fact that, so far as this particular question is concerned, the exact sciences have contributed nothing to the development of engineering theory as it is now understood and employed.

In the science of hydrodynamics, the exact methods of the mathematician lead to conclusions which are known to be so far from the truth that engineers can never rely upon them; and it is just as true to-day as it was in the time of Galileo, that our calculations must of necessity be based upon the observed results of experiment.

But if science has failed to explain the natural laws which govern the flow of water, the labours of hydraulicians have added greatly to the number and to the accuracy of experimental observations. To this work of patient observation and measurement, it must be confessed that very little has been contributed from our own country, but a great deal has been done, and is still being done, in France, Germany, and America. The tables that were calculated from the older formulæ are scarcely accurate enough to satisfy modern requirements, and may be rendered more exact by a careful collation of the evidences which have been recorded in the results of these later experiments; while it is also possible to convey those results in a form that would be more useful for practical purposes than the somewhat elaborate formulæ by which they have been expressed.

If we turn to another series of problems, which are related, not to the flow, but to the pressure of water, the state of our existing knowledge and the methods of present practice must be very differently described. Here the calculations of the engineer rest upon the surest foundations, for the principles of hydrostatics have long ago been established, and can never be changed. The calculations, therefore, can be made with the utmost confidence, upon purely rational principles, and with the certainty that they will not be vitiated by any errors of experimental observation.

The engineer will be concerned mainly with the calculable *effects* of hydraulic pressure upon the various structures or machines that may form part of his great constructive works; and although the principles of hydrostatics are in themselves exceedingly simple, yet their application to his own practical purposes will be as complex as are the varying forms which his structures may assume, and the varying conditions under which they have to be executed.

The works that he is engaged upon may be designed for the collection and storage of water, or for its conveyance in aqueducts and open channels, or through tunnels, culverts, and pipe-lines—either for the supply of towns, the drainage of towns and districts, or the irrigation of lands. In other cases they may relate to the establishment of water-courses for purposes of navigation, the correction of rivers, and the prevention of floods—or perhaps to the utilization of water-power, or the hydraulic distribution of power from central stations. And the principles of hydraulic engineering will have their application over a still wider field, which must include many works of harbour-construction, the sinking of foundations in deep water, and the transport of heavy loads upon floating pontoons.

In connection with all these matters, the engineer will have to deal with the effects of hydraulic pressure under a great variety of conditions. Sometimes the calculations will be very simple, but in other cases the problem will require a very close and careful examination.

In the following chapters, some of the most noteworthy of these problems will be discussed from a purely practical point of view. The whole subject will naturally come under two main divisions, and the first part of the book will be devoted to the static pressure of water, or fluid pressure in general, and the calculation of its effects in engineering structures.

The very different class of questions relating to the flow of water will be reserved for the second part of the book, in which it will also be necessary to consider those variations of pressure which may be due to the flow of water, or to changes in its velocity.

PART I.

FLUID PRESSURE, AND THE CALCULATION OF ITS EFFECTS IN ENGINEERING STRUCTURES

CHAPTER I.

PRACTICAL CONCEPTIONS OF ELEMENTARY PRINCIPLES.

Art. 1. Fluid Pressure.—In beginning a study of hydrostatics, the reader often finds that he is expected to frame his mental conceptions of fluid pressure on the somewhat arbitrary lines of certain propositions which follow in logical sequence from some definition of the nature of a fluid. These propositions do not afford much help to the practical engineer, who has already formed his own ideas of the subject in a slightly different shape; and if he happens to read that, at any point in the internal mass of a fluid, the pressure acts equally in all directions, and in all directions at the same time, he may possibly find some difficulty in attaching any definite meaning to the statement.

But if the essential facts in regard to the action, the direction, the magnitude, of fluid pressure are stated in his own language, he will always recognize them as familiar truths, and they will be to him quite as self-evident as any axioms that could be laid down.

In general he is probably accustomed to regard the pressure as a distributed force acting, not in all directions, but in one definite direction, upon some solid body with which the fluid comes in contact.

If a square box, like that sketched in Fig. 14, is filled with high-pressure steam, he sees at once that the force acts upon the underside of the lid in a direction vertically upwards, on the floor vertically downwards, and on the sides of the box in horizontal directions tending to force them outwards to the right and to the left. In each case the direction in which the pressure acts is at right angles to the plane of the boundary-wall; and if the box were placed in a tilted position at any angle of inclination, the direction of all the pressures would, of course, be altered accordingly, the pressure upon each face being *always* at right angles to the plane surface; so that a plane surface presented to the contact of the steam in any sort of position would experience the action of the steam-pressure as a force at right angles to its own plane—and exactly at right angles, because we can hardly conceive of any tangential or shearing force between the fluid and the boundary-wall, unless we are considering the special case of a fluid in rapid motion.

It is quite easy to see, therefore, that if we imagine a plane *section* to be taken through the internal mass of the fluid, we shall find a certain fluid stress across the plane of that section, or a pressure between fluid and fluid; and that this stress or pressure will *always* be at right angles to the imaginary plane, whatever position or inclination we may choose for it.

It may be useful to realize this idea of an internal stress or pressure, for we shall sometimes have to consider it as one of the forces concerned in the equilibrium of structures.

Art 2. Fluid Pressure of Uniform Intensity.—The intensity of fluid pressure is usually expressed in pounds per square inch or per square foot, and is commonly measured for practical purposes by certain instruments which need not here be described.

When the pressure has a uniform intensity, p, upon every square inch of a plane surface whose area is A, it is a very simple matter to calculate the whole force, or the whole pressure P exerted by the fluid upon that area in a direction at right angles to the surface; for it will be simply $P = pA$, and the pressure will of course have the same value whatever may be the position or inclination of the surface.

For the ordinary calculations of engineering, the pressure of steam, of compressed air, or of any other gas, would usually be treated as a pressure of uniform intensity; and if a pressure-gauge or indicator were used to measure the intensity of steam-pressure in the square box of Fig. 14, the engineer would expect the gauge to register the same pressure whether it were applied at the top, at the bottom, or on either side of the box. In this case it is obvious that the internal pressures on the opposite sides of this closed vessel must exactly balance each other, so that on the whole the internal pressure would have no tendency to move the box in either direction.

If a small vessel is charged with hydraulic pressure under a great head, we shall introduce no serious error by assuming the pressure, in this case also, to have a uniform intensity, although in strict truth the assumption would be incorrect. To obtain a measurement of the error that would thus be involved, we might take a cubical box measuring 12 inches on each side, and placing it upon a weighing machine, let the weight of the empty box be registered. Then after charging the vessel with water under an accumulator pressure of 1000 lbs. per square inch, we should of course find the weighing machine to register a greater load than before, the difference being exactly the weight of the cubic foot of water, or say 62½ lbs. But the box is now acted upon by three vertical forces (besides its own weight), and these forces balance each other. The upward exterior pressure or reaction at the base of the box is 62½ lbs. as registered by the weighing machine; and it follows that the internal fluid pressure acting vertically downwards upon its floor must be greater than that which acts vertically upwards upon the lid, by exactly 62½ lbs. The force exerted by the fluid upon each face of the cube under the assumed uniform pressure of 1000 lbs. per square inch, would amount to 144,000 lbs. But if that figure represents exactly the lifting force upon the lid, the floor must evidently be subjected to a downward pressure of 144,062 lbs., or to a fluid pressure of 1000·43 lbs. per square inch. The difference, in such a case as this, is too small to be worth considering, unless a high degree of accuracy is required in the calculations, and in practice we should seldom need to take account of so small a percentage of error.

As we could nowhere find a fluid without weight, it follows that we could hardly ever find a fluid pressure of really uniform intensity; but when the variations are so slight, we can neglect them in practice, and treat some of our problems as though p were a constant quantity; and when the surface is a horizontal plane, the pressure on that surface will in general be quite uniform.

Art. 3. Hydrostatic Pressure of Varying Intensity.—A different set of problems will come before us when we are dealing with standing water, open to the air at its upper surface, or subjected *only* to the pressure which is due to its own weight; for here the pressure-intensity p must be considered as a varying quantity proportional to the vertical head h. In other words, if we conceive the whole mass of the standing water to be divided into a number of horizontal layers, the fluid pressure in each layer will be proportional to the depth at which it lies below the open surface.

If this statement needed any demonstration, it would again be sufficient to refer to those commonplace facts which are familiar to the mind of every practical engineer. Thus, taking off the lid from the cubical vessel before referred to, and placing the open box upon a weighing machine, we may fill the vessel up to the brim with cold water, at a temperature of say 40° Fahr.; and we shall then find that, after deducting the weight of the empty vessel, the net weight of the cubic foot of water contained in it is just 62·425 lbs. As the fluid is now exerting *no* lifting force, and as the horizontal pressure on the sides of the cube cannot possibly affect the weighing machine, it follows that the load of 62·425 lbs., registered by the weighing machine, is simply the distributed downward pressure of the fluid upon the flat bottom of the vessel. In this horizontal layer of water, therefore, lying close to the bottom at a depth of 1 foot below the open surface, the pressure p is 62·425 lbs. on the square foot, or 0·433 lb. per square inch; and the same pressure will act in a horizontal direction upon the vertical sides of the vessel surrounding the layer.

To find the fluid pressure p in a layer lying at any greater depth, we may carry up the vertical sides of the box, and then fill it to a greater height h and weigh it again. If we fill it to a height of 10 feet, for example, the net load registered by the weighing machine will be the fluid pressure per square foot in a layer 10 feet below the open surface.

In the case of water, which is a practically incompressible fluid, the density is not altered by any increase of pressure within ordinary limits, and for this reason the net load registered by the weighing machine will be exactly ten times as great as in the first instance, or exactly 624·25 lbs., showing a fluid pressure of 4·33 lbs. per square inch.

Obviously, therefore, we have the following general rule for the hydrostatic pressure p at any point whose vertical depth below the open surface is denoted by h, viz.:—

$$p = \gamma h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where γ is the weight of the incompressible fluid per unit of volume.

Measuring h in feet, and inserting for γ its value for fresh water at the temperature of greatest density, we have

$$\left.\begin{array}{r} 62{\cdot}425h = p \text{ in pounds per square foot} \\ 0{\cdot}433h = p \text{ in pounds per square inch} \end{array}\right\} 1a$$

and of course this pressure p is the same, at whatever angle of inclination the boundary-wall may present its surface to the contact of the fluid; while the direction of the pressure is always at right angles to that surface.

At the beginning of this article, we assumed that the standing water would be open *to the air* at its upper surface. At its upper surface, therefore, it would be exposed to the barometric pressure; and if the water surrounds the sides of a condenser we shall have to add the barometric pressure to the value of p as above determined; but in most other cases we may leave the atmospheric pressure out of account, because the structures we are concerned with are themselves surrounded by atmospheric pressure.

The forces, upsetting moments, and internal stresses, caused by the action of fluid pressure upon solid bodies, will of course be determined by the ordinary principles of static equilibrium.

CHAPTER II.

THE ACTION OF FLUID PRESSURE OF UNIFORM INTENSITY.

Art. 4. Centre of Action.—In considering the equilibrium of any structure under the action of fluid pressure, we shall always have to calculate the magnitude of that distributed force, and to find the position of its centre of action. When the pressure has a varying intensity, the calculations will be quite different from those which can be applied to a pressure of uniform intensity; and we may first confine our attention to the latter, assuming the pressure p to be a constant quantity.

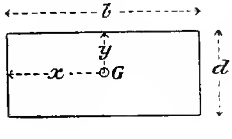
If the surface of a level floor is covered with a load spread evenly, or unevenly, over it, the line of action of the distributed downward pressure must pass through the centre of gravity *of the load* in every case; and if the load is *uniformly* distributed, the centre of action will coincide with what is called the centre of gravity *of the plane surface.*

It is sufficiently obvious that, if we substitute for the level floor a plane surface of the same figure inclined at any angle with the horizon, and for the load a fluid pressure acting at right angles to that surface, the centre of action must still have the same position in the surface, and will coincide with its so-called centre of gravity if the pressure-intensity p is uniform.

For surfaces of regular geometrical outline we may here note down, for future reference, the following values of the superficial area A, and of the co-ordinates x and y, which fix the position of the centre of gravity.

1. For any *rectangle*, whose breadth is b and depth d, we have, of course, very simply—

FIG. I



$$A = bd$$

$$x = \frac{b}{2}, \text{ and } y = \frac{d}{2}$$

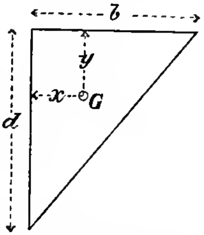
2. In a *right-angled triangle,* whose base and height are denoted by b and d, the area will be—

$$A = \frac{bd}{2}$$

while the ordinates measured from the square corner in Fig. 2 will be—

$$x = \frac{b}{3} \text{ and } y = \frac{d}{3}$$

FIG. 2



3. In *any triangle, ABC,* the perpendicular distance of the centre of gravity G from *any* axis OX or OY in Fig. 3, will always be the arithmetical mean of the three ordinates measured from the same axis to the three corners A, B, and C. Thus to find the "centre of effort" of a cutter's jib, which is the same thing as the centre of action of a uniform wind-pressure, we have for its height above the water-line OX—

$$y = \frac{AA_2 + BB_2 + CC_2}{3}$$

or for its horizontal distance x from a vertical line OY—

$$x = \frac{AA_1 + BB_1 + CC_1}{3}$$

The moment of the wind-pressure pA (upon the whole sail) about any axis OX is, of course, the product pAy.

FIG. 3

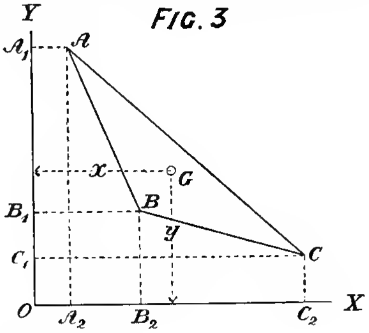
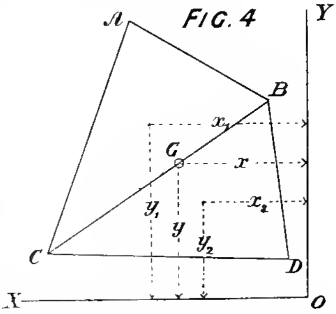


FIG. 4



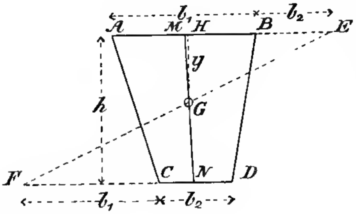
4. In any *quadrilateral, ABCD* (Fig. 4), the area A will be the sum of the two areas A_1 and A_2 of the two triangles ABC and BCD; and finding for these triangles the values of the co-ordinates x_1, y_1 and x_2, y_2 to the centre of gravity of each, we can easily calculate the co-ordinates x and y to the centre of gravity G of the whole quadrilateral by summating the moments of the two triangles. Thus if Fig. 4 represents the mainsail of a cutter, the wind-pressure on the whole

sail will be $P = pA = p(A_1 + A_2)$; while the moment about the water-line OX will be $pAy = p(A_1y_1 + A_2y_2)$; and the height of the centre of effort G above the water-line will therefore be—

$$y = \frac{A_1y_1 + A_2y_2}{A_1 + A_2}$$

$$\text{also } x = \frac{A_1x_1 + A_2x_2}{A_1 + A_2}$$

FIG. 5



5. The *trapezium* $ABCD$ of Fig. 5 may be treated in the same way; but as the sides AB and CD are parallel, we can denote by h the perpendicular distance between them; and then if b_1 and b_2 represent the widths AB and CD respectively, the ordinate HG may be expressed by—

$$y = \frac{h}{3} \cdot \frac{b_1 + 2b_2}{b_1 + b_2}$$

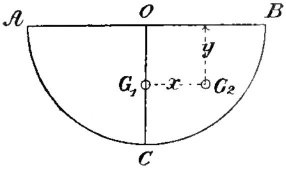
while the area of the figure is—

$$A = h \cdot \frac{b_1 + b_2}{2}$$

The centre of gravity G is perhaps most easily found by graphic construction. Thus bisecting AB and CD in M and N, the point G must lie in the straight line MN. Then producing AB and DC to E and F, we can set off the lengths AE and DF each equal to $b_1 + b_2$; and a straight line EF will then intersect the line MN in the required point G.

6. In every *regular polygon*, as in every circle, the centre of gravity will of course coincide with the centre of the figure.

FIG. 6



The *semicircular area* ACB in Fig. 6 will evidently have its centre of gravity G_1 upon the vertical line OC drawn through the centre of the circle, and the ordinate OG_1 will have the value—

$$y = \frac{4r}{3\pi} = 0{\cdot}4244r$$

where r is the radius OC. The area of the semicircle is—

$$A = \frac{r^2\pi}{2} = d^2\frac{\pi}{8}$$

where d is the diameter; so that under a uniform pressure p, the moment of the distributed force about the axis AB will be—

$$pAy = \tfrac{2}{3}pr^3 = \frac{pd^3}{12}$$

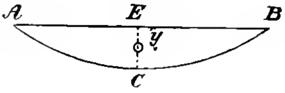
The *quadrant OBC* in the same diagram will have its centre of gravity at the point G_2, whose co-ordinates measured from the axes OC and OB will evidently be—

$$x = y = \frac{4r}{3\pi} = 0{\cdot}4244r$$

while the area of the quadrant, and its moment about either axis, will have one-half of the values given for the semicircle.

7. The *parabolic segment ACB* in Fig. 7 has an area equal to two-thirds of the rectangle $AB \times CE$; or if b denotes the width and h the central height of the segment, then—

FIG. 7



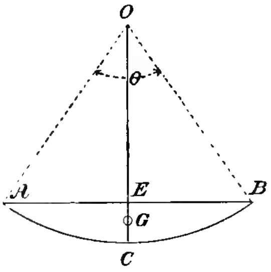
$$A = \tfrac{2}{3}bh$$

while the ordinate EG, to its centre of gravity G, is—

$$y = \tfrac{2}{5}h$$

8. The *segment of a circle* differs very little from the parabolic segment so long as the ratio of h to b is a small fraction, as it is, for example, in the invert of a culvert of ordinary design. To find its exact area A, we may subtract the area A_2 of the triangle OAB, from the area A_1 of the sector $OACB$ in Fig. 8. To make the calculation, the angle θ may first be found from a table of sines by means of the expression—

FIG. 8



$$\frac{AE}{OA} = \sin\frac{\theta}{2}$$

and then we have—

$$A_1 = \text{area of whole circle} \times \frac{\theta}{360^\circ}$$

$$A_2 = AE \times OE$$

$$A\ = A_1 - A_2$$

If the height EC is denoted by h, the ordinate EG, in segments less than a semicircle, will always lie between $0{\cdot}4h$ and $0{\cdot}424h$. The exact position of G is given by—

$$OG = \frac{b^3}{12\mathrm{A}}$$

in which b denotes the width AB in the diagram.

Art. 5. The Pressure directly opposed by Contrary Forces or Resistances.—In the plunger of a feed-pump and the ram of a hydraulic cylinder, we have examples of a *direct* opposition between the normal pressure at the base of the ram or plunger and the external resistances or forces; while the piston of a steam-engine works against an external resistance which is transmitted through the piston-rod to the centre of gravity of the plane circular surface, and is therefore directly opposed to the "resultant" of the fluid pressure. In the same way a pressure of uniform intensity acting normally upon any one of the plane surfaces, illustrated in the last article, may be directly opposed by an equal and opposite force or resistance $\mathrm{P} = p\mathrm{A}$ applied at the point G in each figure.

More frequently, however, we have to consider structures in which the equilibrium is maintained by an *indirect* opposition of resistances to the normal fluid pressure.

Art. 6. Resolution of the Fluid Pressure into Components whose Direction is known or mechanically determined.—The method described in this article will be found very useful for the solution of a large number of questions, for it very often happens that our calculations are concerned only with a particular component of the fluid pressure.

Suppose the hollow trunk, whose longitudinal section is sketched in Fig. 9, to have a square or rectangular cross-section as represented in Fig. 9a, and to be fitted with a square or rectangular piston sliding freely in the trunk. Let the piston

FIG. 9

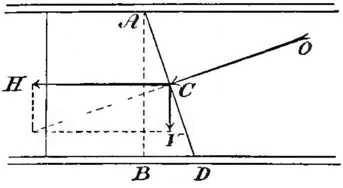
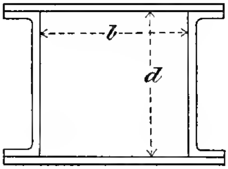


FIG. 9A



be formed with an oblique face AD which is presented to the fluid pressure, and let the angle contained between the plane AD and the normal AB be denoted by θ. Then the actual area of the plane surface AD exposed to fluid pressure will be $S = b \times d \sec \theta$; and the pressure acting at right angles to this surface will be a uniformly distributed force having the magnitude $pS = pbd \sec \theta$. This force may be represented by the line OC meeting the plane at its centre of gravity C and at right angles to AD, and will be resolved into the components CH and CV, of which CH represents the force acting in the direction of the axis of the trunk. We have therefore—

$$CH = pS \cos \theta = pbd \sec \theta \cos \theta = pbd$$

In other words, the force by which the piston is impelled along the trunk in the direction of its axis, is independent of the angle θ; and is always equal to the pressure p multiplied by the area of the rectangle bd of Fig. 9a, which is a projection of the inclined face of the piston in a plane at right angles to the axis of the trunk.

In like manner the vertical component will be

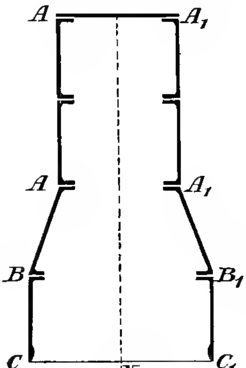
$$CV = pS \sin \theta = pbd \sec \theta \sin \theta = pbd \tan \theta$$

in which expression the quantity $d \tan \theta$ is equal to the horizontal length BD subtended by the inclined surface. The *vertical* component, therefore, is equal to the pressure p multiplied by the area of a *horizontal* projection of the inclined plane; and it is obvious that whatever inclination the plane AD may have to the vertical plane AB, between 0 and 90°, the horizontal and the vertical components will always be measured by the areas of the projections of that surface on planes which are respectively at right angles to the directions of those component forces.

The force CH may evidently be determined by the same graphic method when the surface exposed to fluid pressure is made up of any number of elements or planes inclined at different angles to the axis, or when it consists of any cylindrical, conical, spherical, or other surface or surfaces of single or double curvature; for in all such cases we may conceive the whole surface to be made up of a large number of such elements, each having its own inclination to the axis CH, and each forming the oblique base of a prism whose axis is parallel to CH, like the prism sketched in Figs. 9 and 9a.

In general the action of fluid pressure in engineering structures and machines is equilibrated by forces, stresses, or resistances applied in certain definite directions, and the magnitude of the component which has thus to be opposed can be readily found by this method. A few examples will illustrate its application.

FIG. 10



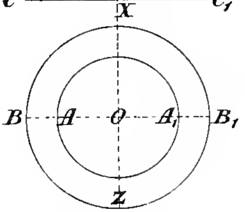


FIG. 10A

Example 1.—The bridge cylinder sketched in section in Fig. 10 has a diameter AA_1 which is increased to the diameter BB_1 below the conical junction ring. The pneumatic pressure p, employed in the sinking of the cylinder, will act everywhere at right angles to the internal surface of cone and cylinder; but in connection with the requisite weighting of the cylinder, it may be desirable to calculate the vertical component of the fluid pressure, or the lifting force exerted by the compressed air.

As the direction of the required force is vertical, draw a horizontal projection or inverted plan of the surfaces as in Fig. 10*a*, where AA_1 represents the cover of the cylinder, while the annular area between the circles AA_1 and BB_1 will be the horizontal projection of the conical shoulder. This annular area multiplied by p will give us the vertical component of the pneumatic pressure acting upon the conical surface, and this vertical force added to the direct upward pressure upon the cover of the cylinder, will give us the total lifting force, which is therefore equal to the pressure p multiplied by the whole area of the circle BB_1 in Fig. 10*a*.

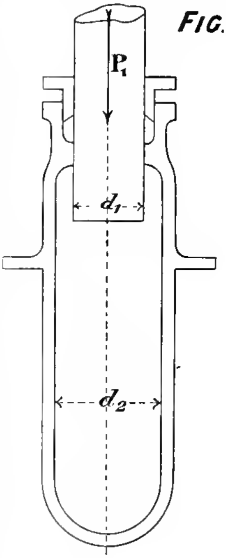
Example 2.—In the same bridge cylinder it may be necessary to calculate the resistance of the cylinder itself to the bursting pressure, and the requisite strength of the bolts by which the segments of the ring BC are united at their vertical joints. Suppose then that the diametral section BB_1 in Fig. 10*a*, at right angles to OX, passes through a pair of these joints and between the internal flanges by which the segments are united. At each of the joints B and B_1, the bolts have to resist by their direct tensile strength, a force acting in a direction tangential to the circle and parallel to OX, tending to separate the semicircle BXB_1 from the semicircle BZB_1; while the magnitude of that force will be found by summating all the components (in the direction OX) of the radial pressures acting upon the interior surface of the cylinder BXB_1 over the height BC. The rectangle BB_1C_1C in Fig. 10 is a projection of that surface on a plane at right angles to OX, and therefore the area of that rectangle multiplied by the pressure p will give us the force which is exerted in the direction OX, and which is equally divided between the two joints, so that one-half of the force thus found will have to be borne in direct tension by the row of bolts uniting the joint *BC*.

The tangential stress in the metal of the cylinder itself is determined in the same way by taking any diametral section through the metal of the cylinder; and it is obvious that the sum of the tensile stresses in the metal cut through by the section is equal and opposite to the internal stress of the fluid acting across the plane of the same section.

In this example, as well as in the previous one, it is assumed that the cylinder *BC* and the conical junction-ring are standing above the water-level. If they were submerged, the tensile stress in the cylinder and the lifting force due to the internal pneumatic pressure would both be reduced by the contrary action of the external fluid pressure; but this point may be left for discussion at a later period.

EXAMPLE 3.—The vessel shown in section in Fig. 11 may be taken to represent either the barrel of a plunger-pump or the cylinder of a hydraulic engine or ram; and it may be required to find, in either case, the vertical forces and stresses acting upon the vessel and upon the framing to which it is attached. The circle drawn in Fig. 11*a* to the diameter d_2, which is the internal diameter of the cylinder, may be taken as a ground plan or horizontal projection of the vessel's bottom, which is visible from above; or if we regard the figure as an inverted plan, the inner circle will represent the base of the ram, or the circular opening in the stuffing box; so that the annular space between the two circles will be a horizontal projection of the internal shoulder which is visible from below.

FIG. 11



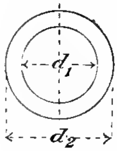
The vertical force exerted by the fluid pressure upon the vessel as a whole, and borne by the framing which supports the outer flanges, will be the algebraical sum of all the vertical components, or the difference between the lifting pressure at the shoulder and the downward pressure upon the bottom of the cylinder. Therefore, subtracting the negative area of the annular space in Fig. 11*a* from the positive area of the larger circle, we have remaining the area of the inner circle, and this area multiplied by p will give us the downward pressure upon the engine-framing. This force is obviously equal to the load P_1 upon the ram, and is independent of the diameter d_2. At the same time it will be at once seen that the vertical tensile stress in the sides of the cylinder below the framing will have the greater value—

$$P_2 = P_1 \frac{d_2^2}{d_1^2}$$

while above the framing it will be—

$$P_3 = P_2 - P_1 = P_1 \frac{d_2^2 - d_1^2}{d_1^2}$$

FIG. 11A



Example 4.—Suppose a cast-iron pipe to be formed as a bend curving through 180° as sketched in Fig. 12, and to be connected at each end to the straight vertical pipes by an ordinary flange-joint. It is required to find the stress in the flange-bolts when the whole line of pipe is charged with a hydraulic pressure p.

The vertical lifting force which must be resisted by these bolts will evidently be the algebraical sum of all the vertical components of the radial pressures acting upon all the elements of area of the internal surface. In the straight vertical pipes all these vertical components will be zero, because normals drawn to all the elements of internal surface will be horizontal; but in the bend these normals will be inclined to the horizontal, some in one direction and some in the opposite direction. The dotted centre-line ACB in Fig. 12, if it were traced upon the sides of the pipe itself, would lie in vertical planes forming tangents to the pipe at all cross-sections of the bend; and we may conceive the centre-line as dividing the pipe into two troughs of semicircular section. The upper trough will contain all those elements of surface which are exposed to pressure having a vertical component in the upward (positive) direction; while the lower trough will contain all those parts of the surface on which the pressure has a downward vertical component. The horizontal projections of these surfaces are easily traced in the sectional plan of Fig. 12a. The figure $ABFB_1A_1E$ is the projection of the positive area, while the figure $ABHB_1A_1G$ is the projection of the negative area; and subtracting the latter from the former, we have remaining the areas of the two circles which represent the open ends of the bend.

FIG. 12

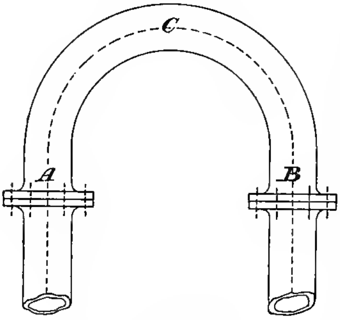
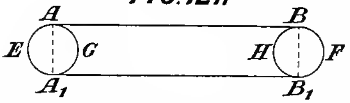


FIG. 12A



The lifting force which has to be resisted at *each* joint is therefore $pd^2\frac{\pi}{4}$, where p is the fluid pressure and d the internal diameter of the pipe.

The same result might have been obtained still more simply by taking an imaginary section across the bend at the plane AB, and regarding the bend *with its contained water*, as forming a body whose equilibrium is to be considered. For the stress or pressure of fluid upon fluid, across this plane, must be balanced by the tensile resistance of the bolts.

CHAPTER III.

CYLINDERS, PIPES, AND BENDS UNDER UNIFORM PRESSURE.

Art. 7. Tangential Stress or Hoop-tension.—In determining the strength of any steam boiler, hydraulic cylinder, or cylindrical pressure-pipe, the engineer will naturally begin by calculating the "hoop-tension" as it is called, or, in other words, the tensile stress which is transmitted round the circumference of the cylindrical shell, and which acts everywhere in a direction tangential to the circle. It will presently be seen that this is not, by any means, to be regarded as the only stress that we have to deal with; but it is the simplest part of the problem, and may conveniently be taken first.

Let d be the internal diameter of the cylinder whose cross-section is drawn in Fig. 13a, and through the cylinder take any diametral section such as that shown by the line AA_1 dividing the vessel into two semi-cylindrical troughs. Also in Fig. 13, which is a longitudinal section on the same plane, let AB represent a unit of length, say 1 inch lineal, of the cylindrical vessel, so that the rectangle AA_1BB_1 is the projection of the internal surface of one of the troughs on a length of 1 inch. We have already seen that when the vessel is subjected to internal pressure, the force which tends to separate the two troughs from each other is equal to the pressure p multiplied by the area of the rectangle, and if the length AB is 1 inch, the force will be simply pd.

If, then, we may assume that, on each lineal inch of the cylinder, this force is resisted by the direct tension in the same lineal inch of the metal shell, whose thickness is denoted by t, we should have for the mean tensile stress per square inch of metal the well-known expression for the hoop tension, or—

$$\text{Mean } f = \frac{pd}{2t} \quad . \quad . \quad . \qquad . \quad . \quad . \quad (1)$$

We can make this assumption without any doubt in the case of a long pipe of uniform diameter and uniform thickness if the metal is fairly homogeneous. In the case of a long boiler or cylinder with closed ends, we could make the same assumption with *safety*, and it would be nearly true for a lineal inch of shell taken at the central portion of the length. But near the ends of the boiler the force would be resisted in part by the tensile strength of the cover, and the hoop-tension in the shell would be somewhat diminished. If, however, we went on to calculate the exact distribution of the stress between the ends and the adjacent shell, by reference to their relative extensibilities, the calculation might be seriously interfered with by change of temperature, initial stress, and unequal elasticity. In most cases, therefore, engineers will prefer to leave out of account any assistance that could be derived from the strength of the end-covers, and will generally design the shell of sufficient strength throughout to resist the hoop-tension as found by the usual formula.

Apart from this matter, however, there are two other points in connection with the hoop-tension which may sometimes have to be considered. If the cylinder is surrounded by another fluid under the pressure p_1, its effect on each lineal inch would be to produce a compressive force $p_1 d_1$, in which d_1 is the external diameter of the cylinder; so that the compressive stress in a boiler-flue, for example, would be mean $f = \frac{p_1 d_1}{2t}$. And if the cylinder is subjected at the same time to an internal and an external pressure, the hoop-tension would have for its mean value—

$$\text{Mean } f = \frac{pd - p_1 d_1}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

FIG. 13

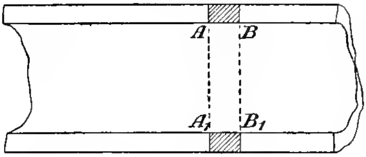
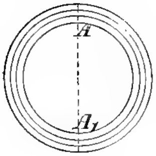


FIG. 13A



These expressions for the *mean* value of the hoop-tension per square inch of metal section will be equally true whether the shell is very thin or very thick; and in the case of a thin cylinder, we may assume that the stress will be distributed almost uniformly over all parts of the metal section, so that we may take the formula as giving practically the maximum tensile stress in the metal.

But when the cylinder is made of great strength to resist very high pressures, as in the case of many hydraulic cylinders, the thickness may become so great, relatively to the diameter, that the stress is very unequally distributed, and the maximum stress may then be considerably greater than the mean value as above given. If we imagine the cross-section to be divided into a number of concentric rings, as shown in Fig. 13*a*, the stress will be most severe upon the innermost ring, and least upon the outermost ring.

The formula that has generally been proposed by mathematicians for calculating the maximum stress is a little too complex for ready use, owing to the fact that it takes into account the external pressure p_1; but in most cases this will represent only the atmospheric pressure which, for practical purposes, may be neglected, and we then have the following values for the maximum stress in terms of the mean stress:—

Thickness in Terms of Internal Diameter, or $\frac{t}{d}$.

	$\frac{1}{100}$	$\frac{1}{80}$	$\frac{1}{60}$	$\frac{1}{50}$	$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
$\frac{\text{Max. } f}{\text{Mean } f} =$	1·010	1·013	1·017	1·020	1·026	1·034	1·052	1·109	1·139	1·190	1·233	1·300	1·417	1·667	2·500

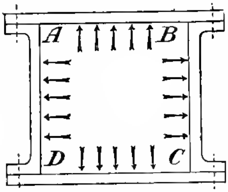
These figures, however, are calculated on the assumption that the cylinder has been manufactured in such a manner as to throw no initial stress upon the concentric rings. It is well known that if the outer rings are subjected to initial tension, throwing the inner rings into initial compression, the maximum tensile stress in the latter, due to internal fluid pressure, will be correspondingly reduced; and by this means the hoop-tension in a thick cylinder may be more evenly distributed over the whole section. Perhaps, however, this question is of more interest to the gun-maker than to the hydraulic engineer, and need not here be considered in detail.

Art. 8. Axial Fluid Stress.—If pressure is to be maintained in any vessel, the fluid stress or pressure must be resisted in *all* directions. In the last article we were considering only the stresses in the transverse plane; but, in addition to these, there is the longitudinal or axial stress, which must be resisted in some way.

To picture the action of fluid stress in the three principal planes, we might turn for a moment to the square box whose cross-section is sketched in Fig. 14; and if we regard it as the cross-section of an endless trunk, we may be content to consider only the stresses which act in the plane of the paper, the opposite pressures on the two vertical sides AD and BC being transmitted through the plates AB and DC, while the vertical pressures on the horizontal surfaces are transmitted in the same way through the vertical sides in direct tension. Thus if d were the diameter of the trunk, we should have in each of the four sides the circumferential tension $f = \frac{pd}{2t}$.

But if we regard the figure as representing the cross-section of a closed cubical vessel, we must consider also the pressure upon the square surface $ABCD$, and this pressure will be transmitted through the *four* sides of the box—through the sides AB and DC as much as through the sides AD and BC. And as each of the three pairs of faces is similarly situated, we must conclude that every one of the six faces of the cube would be subjected to a pair of conjugate stresses, whose directions are parallel to the sides of the square. Also the *average* value of these stresses must be, for each face of the cube, $f_1 = \frac{pd}{4t}$. A close examination of the elastic extensions would very likely show us that the stress is not quite uniform, and would certainly show us that, in the case of a very long trunk of square section, the circumferential stress would be greater in the middle of the length than at the ends, and would approach to the value of f above given; but the longitudinal or axial stress must obviously have the value $f_1 = \frac{pd}{4t}$, and would be uniformly distributed, or very nearly so.

FIG. 14



If we now turn to the case of a cylindrical vessel with closed ends, the pressure upon the end-cover will evidently have the value $P = pd^2\frac{\pi}{4}$, in which d is the internal diameter, and the force P must be transmitted through the shell of the cylinder in direct longitudinal tension, which will be quite uniformly distributed around the whole circumference πd. The uniform intensity of longitudinal stress per square inch of metal section will therefore have the value $f_1 = \frac{pd}{4t}$.

Thus, in a closed cylinder, the effects of the axial fluid stress are clearly manifested in the pressure upon the ends, and are satisfactorily accounted for in the longitudinal tensile stress transmitted from cover to cover through the shell. But in a long line of water-main the same axial stress must be transmitted through the fluid as a compressive force. At any transverse section of the pipe we should find this compressive force P, or compressive stress, between fluid and fluid; and at curved portions of the line it will be attended with certain resultant forces which are worthy of careful study, although they seem to have escaped the notice of most hydraulicians. In ordinary practice, it is true that their effects have not often been noticed; but it is impossible to doubt that under certain conditions these forces would attain a very great magnitude, and, if they were not resisted by some means, would tend to displace the pipe laterally. For this reason we may perhaps call them "displacing forces." Their magnitude and direction might be deduced from the internal axial stress itself by the ordinary principles of statical equilibrium, and this would perhaps be the most direct way of treating them; but their action upon the pipe and upon surrounding bodies is really influenced in a somewhat subtle manner by the varying mechanical conditions under which the pipe may be placed in different cases; and in order to get a clear view of the matter, it may be well to determine the displacing forces, in the first instance, by a direct measurement of the known fluid pressures upon the internal surfaces of the curved pipe or bend, and afterwards to consider the several kinds of resistance by which these forces may be opposed.

Art. 9. Displacing Forces in a Pipe-bend of Polygonal Form.— For convenience of manufacture, a bend of riveted steel pipe is often constructed in the form of a polygon instead of following a truly circular curve, and the same thing is sometimes done when the pipes are of cast-iron. In the latter case the short straight pipes forming the sides of the polygon are often united by a double socket-joint, whose faces are inclined at a certain angle, as sketched in Fig. 17, while in other cases the straight lengths may meet each other with a mitre joint, and may be united by flanged ends bolted or riveted together, as sketched diagrammatically in Fig. 15, which may be regarded as the vertical section of an upright bend, and will serve to illustrate the question in its simplest form.

The pipe *AB* in this polygonal bend will be a cylinder whose ends are formed by the inclined planes *EG* and *FH*; and it is

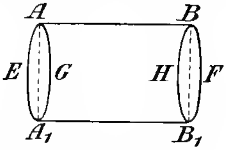


FIG. 15A

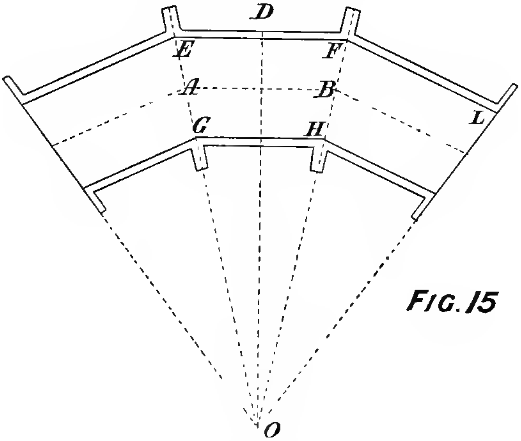


FIG. 15

obvious that the fluid pressure upon the side *EF* will have a greater magnitude than the opposite pressure upon the side *GH*, so that there will be a displacing force in the direction *OD*, which, in this particular case, would tend to *lift* the pipe vertically upwards.

The magnitude of this lifting force will easily be found if we imagine the pipe to be divided into two semi-cylindrical troughs by a section along the centre line *AB*, for the upper trough will then contain all those elements of surface which can be subject to any *upward* force, and *vice versâ.*

Therefore, drawing a projection of the internal surfaces of these troughs on a plane at right angles to *OD*, we have the areas shown in Fig. 15*a*, the entire area *EABF*, etc., being the projection of the upper trough, or the positive area, while the smaller area *GABH*, etc., will be the projection of the lower trough, or the negative area.

Subtracting the negative from the positive area, we have remaining the two ellipses, whose combined areas, multiplied by p, will give us the total lifting force exerted upon the pipe *AB*. If d is the internal diameter of the pipe, and if θ denotes the angle *EOF*, which is bisected by *OD*, then the minor axis of each ellipse will be $d \tan \frac{\theta}{2}$, and the area of each ellipse will be $d^2 \frac{\pi}{4} \tan \frac{\theta}{2}$. The displacing force in the direction *OD* will therefore be $2pd^2 \frac{\pi}{4} \tan \frac{\theta}{2}$; or, if we write P for the quantity $pd^2 \frac{\pi}{4}$, we may express the magnitude of the displacing force by—

$$S = 2P \tan \frac{\theta}{2} \quad . \quad . \qquad . \quad . \quad . \quad (3)$$

In this expression the quantity P is nothing else than the axial fluid stress in the cylindrical pipe of diameter d; and if we consider the fluid stress acting upon the two ends of the pipe *AB* as producing a pair of external forces which have to be equilibrated, we shall find that their resultant has the direction *OD*, and has exactly the magnitude already given by the above formula.

Thus, regarding the fluid contents of the pipe *EFGH* in Fig. 15 as forming *the voussoir of a fluid arch*, the planes *EG* and *FH*, forming the faces of this voussoir, will evidently be of elliptical shape, the major axis *EG* or *FH* being equal to $d \sec \frac{\theta}{2}$, and the fluid stress upon these surfaces will therefore have the magnitude $P \sec \frac{\theta}{2}$ in each case. The lines of action of these forces will, of course, be at right angles to the surfaces *EG* and *FH* respectively, and will intersect on the line *OD*, while their resultant will have the direction *OD* and the magnitude $P \sec \frac{\theta}{2} \times 2 \sin \frac{\theta}{2}$, so that the combination of these forces upon the familiar principles of the parallelogram gives for their resultant the same value—

$$S = 2P \tan \frac{\theta}{2}$$

It follows, then, that one method of equilibrating the displacing forces upon a polygonal bend would be to apply on each side of the polygon *EF*, *FL*, etc., an external force or resistance equal and opposite to the calculated force S. In the case of a buried pipe having such a bend in the horizontal plane, the lateral resistance might be furnished by the reaction of the earth filling or by the concrete surrounding the pipe; while, in the case of an upright bend, the force to be applied to the pipe *AB* would be simply so much dead weight, in which the weight of the pipe itself and its fluid contents would constitute part of the required load. It is also obvious that if we wish to avoid any bending stress in the pipe, the load or reaction ought, strictly speaking, to be applied in exact opposition to the unbalanced fluid pressure, or at two points near the ends of the pipe and opposite the centres of the two elliptical areas.

Another method of preventing any displacement of the polygonal bend would be to apply a sufficient external force at each *angle*, instead of each *side* of the polygon. Thus, if *AB* and *BC*, in Fig. 16, are the centre lines of two straight pipes meeting at an angle *ABC*, the displacing force may evidently be equilibrated by a force or reaction in the direction of the line *FH* which bisects the angle *ABC*. Drawing the sections *MN* and *HK* normal to *AB*, and *HG* normal to *BC*, the angle *KHG*, or θ, will be the supplement of the angle *ABC*, and will be bisected by *HF*. The internal pressures on the opposite sides of the pipe, between the normal sections *NM* and *KH*, will exactly balance each other, but the pressures on all the parts of the

FIG. 16

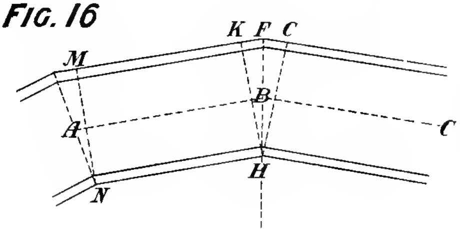
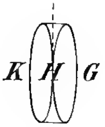


FIG. 16A



internal surface *KFGH* will not balance each other. To find the displacing force acting on this surface in the direction *HF*, we may draw a projection of the surface on a plane at right angles to *HF*, as in Fig. 16*a*; and subtracting as before the negative from the positive projection, we have the two ellipses, whose minor axes *KH* and *HG* will each be equal to $d \sin \frac{\theta}{2}$. The displacing force will accordingly be—

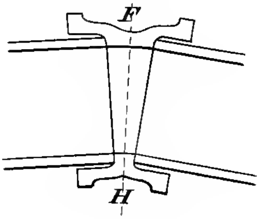
$$N = 2P \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Exactly the same result will, of course, be reached if we regard the cylindrical columns *AB* and *BC* as members of a *fluid polygonal frame* under an axial compressive force in each member, equal to P or $pd^2\frac{\pi}{4}$.

When the straight lengths of pipe are formed with plain ends united by double sockets, whose faces are inclined to each other at the angle θ, as sketched in Fig. 17, there will evidently be no unbalanced force resulting from the internal pressure on the sides of the straight pipes; while the socket itself will be subjected to a displacing force in the direction *HF*, whose magnitude will be independent of the length of the socket, but will be proportional to $\sin \frac{\theta}{2}$ as before.

It seems probable, however, that the fluid would find its way into the joint so far as to take effect over the annular surface at the ends of the plain pipe; and if their external diameter is d_1, the area of each elliptical projection will be $\frac{\pi}{4}d_1^2 \sin \frac{\theta}{2}$; or, if we denote by P_1 the axial pressure taken over the *whole* section of the pipe (including the annular surface), the displacing force will be—

FIG. 17



$$N_1 = 2P_1 \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

EXAMPLE.—Suppose a cast-iron pipe, having an internal diameter of 36 inches, and an external diameter of 39 inches, to be laid in straight lengths, whose centre lines form the tangents to a curve of 50 feet radius. And let the length of each side, *AB*, *BC*, etc., be 5·24 feet, so that the angle θ subtended at the centre of the curve by each side of the polygon is 6°, the internal angles *ABC*, etc., being each equal to 174°.

If this pipe were charged under a hydrostatic head h of 320 feet, the axial fluid stress would be—

$$P = 320 \times 62{\cdot}4 \times 3^2 \times 0{\cdot}7854 \div 2240 = 63 \text{ tons}$$

But, in the case of the socket-joint, the axial stress over the *whole* area of the pipe would be—

$$P_1 = 320 \times 62{\cdot}4 \times 3{\cdot}25^2 \times 0{\cdot}7854 \div 2240 = 73{\cdot}94 \text{ tons}$$

Also we have—

$$\tan\frac{\theta}{2} = 0{\cdot}0524; \ \sin\frac{\theta}{2} = 0{\cdot}05233$$

The displacing force at right angles to each *side* of the polygon, in such a case as Fig. 15, would then be—

$$S = 2 \times 63 \times 0{\cdot}0524 = 6{\cdot}7 \text{ tons}$$

which is equivalent to 1·26 tons per lineal foot of the pipe *AB.*

Or, again, at each *angle* of the same polygon, the radial displacing force would be—

$$N = 2 \times 63 \times 0{\cdot}05233 = 6{\cdot}59 \text{ tons}$$

But, in the case of the double socket illustrated in Fig. 17, if we allow for the pressure upon the annular ends, we shall have—

$$N_1 = 2 \times 73{\cdot}94 \times 0{\cdot}05233 = 7{\cdot}74 \text{ tons}$$

The displacing force would be equilibrated *either* by a force equal and opposite to S acting upon each side, *or* by a force equal and opposite to N acting upon each angle. In this example the difference between N and S is very slight, because θ is a small angle; but if θ were large the difference would be greater. Thus, if θ were 90°, we should have $S = N\sqrt{2}$.

Art. 10. Displacing Forces in a Circular or Segmental Bend.—If a cast-iron pipe is formed as a curved bend, its centre line *ADB* lying in the arc of a circle as sketched in Fig. 18, it is again evident that the pressures on the opposite internal sides of the pipe will not balance each other; for if we conceive the pipe to be divided into two bent troughs by a diametral section passing along the centre line *ADB,* the areas of those trough surfaces will not be equal, nor will their projections be equal; and the sum of all the components of fluid pressure measured in the direction *OD* will be greater than the components measured in the opposite direction.

FIG. 18

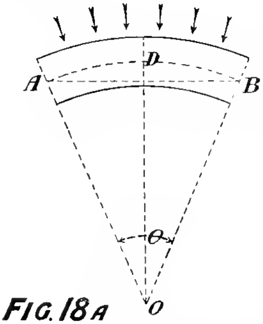


FIG. 18A

Considering the segment ADB as forming either the entire bend, or one of a series of similar segments constituting the bend, let θ denote the angle subtended by the arc, and let the segment be terminated at A and B by the radial plane sections OA and OB. Then, to find the magnitude of the displacing force which acts in the direction OD at right angles to the chord AB, draw a projection of the trough surfaces on a plane parallel to AB, and, subtracting the negative area as before, we have the two ellipses of Fig. 18*a*, in each of which the minor axis is $d \sin \frac{\theta}{2}$.

The combined area of these two ellipses will be $2d^2\frac{\pi}{4} . \sin \frac{\theta}{2}$. The displacing force in the direction OD, or the resultant of the unbalanced pressures on the internal surfaces of the bend AB, will therefore have the magnitude—

$$N = 2pd^2\frac{\pi}{4} . \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$= 2P \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

$$= P \times \frac{\text{chord } AB}{\text{radius } OA} \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

If we proceed to subdivide the segmental arc ADB into a number of indefinitely small and equal arcs, it is evident that by repeating the same calculation we shall find upon every one of those little arcs a displacing force acting upon the pipe in a direction radially outwards (from the centre of curvature O), or at right angles to the curved centre line of the pipe; and on each of these equal elements of the arc, the displacing force will have the same magnitude, so that we have really to consider the force as *a uniformly distributed pressure* acting normally to the arc at all points; and the intensity of this pressure per lineal foot of the pipe is easily to be found.

If θ is indefinitely small, the chord is sensibly equal to the length of the arc, and from the formula (5*b*) above given, it is evident that the distributed radial force or pressure must have for its intensity per lineal unit of arc the value—

$$n = \frac{P}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

so that the pipe-bend would be perfectly equilibrated at all points if, along with the internal fluid pressure, it were subjected to an external pressure uniformly distributed along the arc, and acting normally to the curve in a direction towards the centre *O*, with the intensity $\frac{P}{R}$ per unit of arc-length.

Exactly the same result will be reached if we regard the curved column of water as a *fluid arch* of segmental form in which the compressive stress is known to have the value $P = pd^2\frac{\pi}{4}$; for if the compressive stress in such an arch is to follow the curved "line of pressure," of radius R, we know very well, by the ordinary principles of the equilibrium of arches, that it must be combined with a uniform external pressure (normal to the arch), whose intensity per lineal unit of arc-length must be $n = \frac{P}{R}$, as before.

Example.—In a cast-iron pipe, whose internal diameter is everywhere 36 inches, let the segmental bend *ADB*, subtending an angle of 45°, be formed with a radius *AO* of 20 feet; and let the pipe be subjected to a hydrostatic head of 320 feet.

Here the axial fluid stress P will have the value already found in a previous example, or—

$$P = 320 \times 62{\cdot}4 \times 3^2 \times 0{\cdot}7854 \div 2240 = 63 \text{ tons}$$

$$\text{while } \sin\frac{\theta}{2} = 0{\cdot}38268$$

Therefore the bend will be subjected to a uniformly distributed radial displacing force or pressure, whose intensity per unit of arc-length will be—

$$n = \frac{P}{R} = \frac{63}{20} = 3{\cdot}15 \text{ tons per foot lineal}$$

Upon the whole bend, the resultant of this distributed force will have the direction *OD*, and the magnitude—

$$N = 2P\sin\frac{\theta}{2} = 2 \times 63 \times 0{\cdot}38268 = 48{\cdot}2 \text{ tons}$$

Art. 11. Axial Stress in the Fluid and in the Pipe.—In the preceding articles we have found the displacing forces due to the unbalanced fluid pressures by a direct measurement of those pressures upon the actual internal surfaces of the pipe; but in each case it has been shown that the same result might be more simply obtained by considering the axial stress in the cylindrical column of fluid which fills the pipe. For in every equilibrated arch or polygonal frame there must be a certain adjustment between the so-called "thrust," or compressive axial stress, and the external forces applied to the arch or to the joints of the frame. Of course the same adjustment must always subsist whether this axial stress takes effect wholly in the fluid contents of the pipe, or partly or wholly in the metal of the pipe itself; and therefore if we know in any given case that the pipe is itself subjected to a longitudinal compressive force Q, the effective axial stress P + Q must be substituted for P in calculating the external loads or lateral forces that are required to maintain the equilibrium of the arch or frame.

The formulæ already found for the polygonal bend will then become—

$$S = 2(P + Q) \tan \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{and } N = 2(P + Q) \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (8)$$

while for the segmental bend or arch we shall have—

$$N = 2(P + Q) \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or for the uniformly distributed pressure per lineal unit of arc-length—

$$n = \frac{P + Q}{R} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

EXAMPLE.—Suppose the segmental bend *ADB* already illustrated in Fig. 18 to be fitted at each end with an expansion joint, in which the plain cylindrical end of the pipe slides in the gland of a stuffing-box, as sketched in Fig. 19. Then, if the internal diameter d is uniform throughout the pipe, it is evident that the gross or effective axial stress will be somewhat greater than $pd^2\frac{\pi}{4}$ on account of the fluid pressure upon the annular end of the pipe within the expansion joint.

If d_1 denotes the external diameter of the sliding neck, or the bore of the gland, the area of annular surface will be $(d_1{}^2 - d^2)\frac{\pi}{4}$, and the total axial stress P + Q will be equal to $pd_1{}^2\frac{\pi}{4}$.

For example, let $d = 36$ inches, $d_1 = 39$ inches, R = 20 feet, and $\theta = 45°$, as in the previous example; then for a head of 320 feet, we have as before P = 63 tons, P + Q = 73·94 tons; and the total lifting-force exerted in the direction *OD* will be—

$$N = 2 \times 73{\cdot}94 \times 0{\cdot}38268 = 56{\cdot}59 \text{ tons}$$

or the distributed normal pressure exerted uniformly all along the arc will be—

$$n = \frac{73{\cdot}94}{20} = 3{\cdot}697 \text{ tons per foot lineal}$$

The compressive force Q transmitted through the metal of the pipe will be 10·94 tons, and the stress per square inch of metal section will, at each point, depend on the thickness of the pipe. At the ends it will obviously be equal to p.

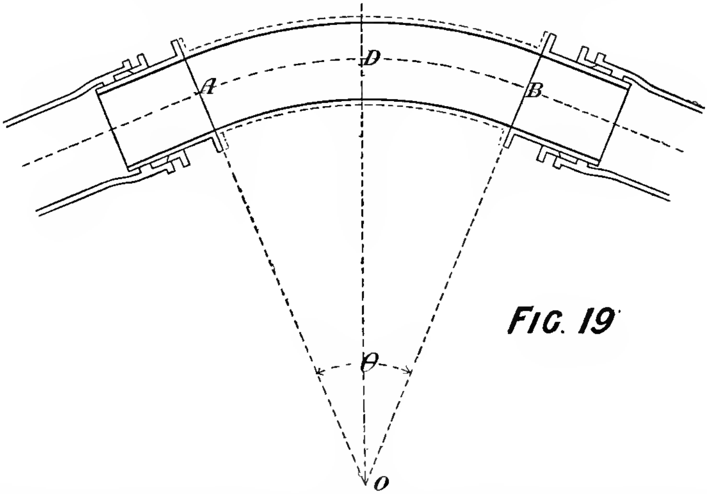


FIG. 19

Art. 12. The Displacing Force resisted by Longitudinal Stress.— The longitudinal stress Q referred to in the last article was a compressive stress of the same character as P; but it needs no demonstration to show that in all cases the effective axial stress, which is really concerned in the equilibrium of the bend, will always be the algebraical sum P + Q, whether Q be positive or negative.

Hence it is easy to see that the displacing forces, which result from unbalanced internal pressures on the sides of the pipe, will be exactly balanced and practically annulled if the pipe itself is subjected to a longitudinal pull − Q, equal and opposite to the axial fluid stress P, so that the sum P + Q of all the stresses over the whole cross-section shall be zero.

In this case the pipe-bend will of course need no lateral support, or load, or reaction, to maintain its equilibrium.

If the displacing forces are to be resisted wholly by this means, it is worthy of note that the requisite tensile stress or force Q will be independent of the radius R and of the angle θ, and must always be equal to P, whether the curvature of the pipe be great or small; also that the metal of the pipe will then be subject to a pair of conjugate stresses, viz. the "hoop-tension" $f=\frac{pd}{2t}$, and at right angles to this a longitudinal stress, whose intensity q is always one-half of the hoop-tension, or $q=\frac{f}{2}$, so that a pipe which is strong enough to resist the hoop-tension is always more than strong enough to resist the longitudinal stress q.

These simple considerations are sufficient in themselves to explain the fact that, in a large number of practical cases, the unbalanced internal pressures exhibit no tendency to produce any lateral displacement of the pipe; but in considering the equilibrium of any pipe-line, it is necessary to inquire whether the tensile stress $-Q$, equal and opposite to P, does or can take place in the metal of the pipe.

It is hardly necessary to remark that, if the bend were closed at both ends, the pressure upon the covers would at once subject the metal to the necessary longitudinal stress, nor to observe that in such a closed vessel the resultant of all the internal pressures must be zero.

Again, in the case of a flexible pipe which is not closed at the ends, a little consideration will show that the conditions are present under which the longitudinal tensile stress could be, and automatically *would* be, adjusted to an exact equality with the axial fluid stress; and in no other way could the equilibrium of such a charged pipe be accounted for.

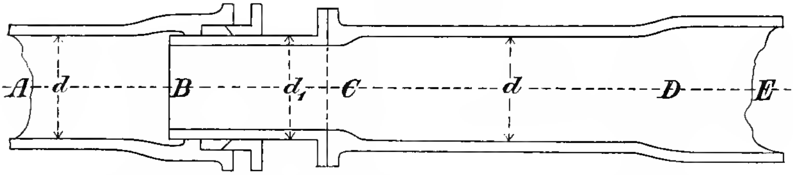
But there are a good many other cases in which the unbalanced internal pressures give rise to very large displacing forces, which cannot possibly be resisted in this way; there are yet other cases in which the magnitude of the tensile stress Q is definitely determined by independent external conditions, or is limited by the tensile strength of the joints; while again there are some others in which it is difficult to estimate the value of that stress with any certainty.

Art. 13. Expansion Joints.—If we introduce a sliding expansion joint at any point in a line of pipe, it is obviously impossible, so long as the joint is really free to slide, that any tensile stress whatever can, at this point, be transmitted through the pipe in a longitudinal direction; and therefore no part of the displacing forces can be resisted by this means.

On the contrary, it has already been shown, in Art. 11, that the axial fluid stress P will be generally augmented in such cases by a *positive* or compressive stress Q due to the hydraulic pressure upon the annular end of the pipe; and the displacing force at any adjacent bend will be proportional to the augmented axial stress $P + Q$, as in the example worked out in that article.

Remembering that the magnitude of the displacing forces is always governed by the *effective* axial stress, which is the algebraical sum of the stresses P and Q, it will be worth while to notice here that, although the stress Q is always a positive quantity at the end of the sliding neck, yet it is quite possible to construct an expansion joint in such a form that the compressive stress shall be confined to a short length of neck, while the remainder of the pipe shall be free from any longitudinal stress, or shall be subjected to longitudinal tension. Thus, for example, if we make the joint in the form which is diagrammatically sketched in Fig. 20, so that the external diameter d_1 of the sliding neck is just equal to the internal diameter d of the pipes *AB* and *CD*, it will easily be seen that neither of these pipes will suffer any longitudinal stress, because the opposite hydraulic pressures upon the two annular ends of the length *BC* will exactly balance each other.

FIG. 20



And if beyond *D* we make the internal diameter of the pipe *DE* greater than d_1, the hydraulic pressure upon the annular surface at *D* must be borne by the pipe *DE* in tension.

In short, the expansion joint fixes, in the most definite manner, the value of the gross effective axial stress which must be transmitted through the pipes in its immediate neighbourhood. For if we regard the pipe *BCDE*, together with its fluid contents, as constituting a body whose circular end at *B* is subjected to the hydraulic pressure, we shall see at once that this external force must be transmitted through the pipes *CD* and *DE*, and in both cases must determine the value of the effective stress P + Q.

If d_1 is the diameter of the sliding neck, or the bore of the gland, the external end pressure will be $P_1 = Pd_1^2\frac{\pi}{4}$; and if d denotes the diameter (as great or as small as we please) of any length, *BC*, *CD*, or *DE*, we shall in all cases have—

$$P + Q = P_1 = pd_1^2\frac{\pi}{4}$$

and as $P = pd^2\frac{\pi}{4}$, the stress Q is determined by the equation—

$$Q = p(d_1^2 - d^2)\frac{\pi}{4} \quad . \quad . \quad . \qquad (11)$$

and may be either positive or negative, according as d is less than or greater than d_1. This equation will apply to the pipe right and left of the expansion joint, *so far as* the pipe is subjected to no other external forces, acting in the longitudinal direction, and would apply to the whole length of the bend *AB* in Fig. 19. In such a vertical bend, therefore, or in any analogous case, the lifting force, or displacing force, will really depend upon the diameter d_1 of the sliding neck; and as the force cannot be resisted by longitudinal tension in the pipe, we have now to consider whether it can be resisted in any other way, or whether it is absolutely necessary to load the pipe with a weight of nearly 4 tons per lineal foot, as calculated in Art. 11.

Art. 14. The Pipe-bend acting as a Girder.—If we suppose the segmental bend sketched in Fig. 19 to be fitted at each end with a cylindrical neck which slides without friction in the gland of the stuffing-box, and that the stuffing-boxes are themselves so fixed to a firm substructure as to resist any displacement, the bend may be regarded as a girder supported at each end by the normal reaction of its bearing in the gland. Between these supports the pipe will be subjected to a uniformly distributed pressure acting normally to the axis, and due to the unbalanced fluid pressure. The girder may be taken as supported, and not fixed at the ends, because the joint could hardly possess fixity of direction, and on this assumption the bending moments can be readily determined.

Finding first (by the usual principles) the external normal forces or reactions at *A* and *B*, we have for the value of each—

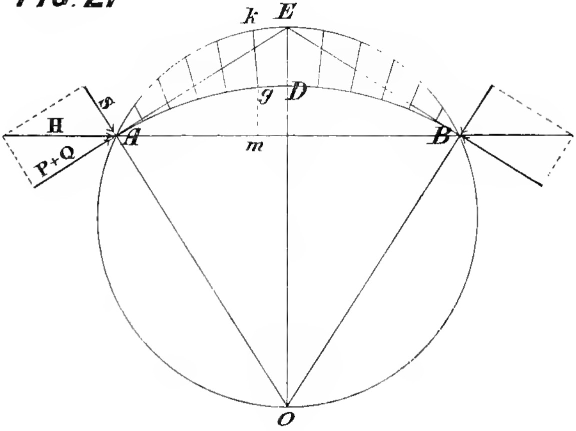
$$S = (P + Q) \tan \frac{\theta}{2} \cdot \qquad . \quad . \quad (12)$$

in which (P + Q) represents the gross axial stress. The maximum bending moment will naturally take effect at the middle of the span, where its value will be expressed by—

$$M_D = (P + Q)R(\sec \frac{\theta}{2} - 1) \qquad (13)$$

To construct a complete diagram of moments for this case, let *ADB*, in Fig. 21, be the neutral axis or centre line of the segmental bend, subtending the angle θ, and let *O* be its centre

FIG. 21



of curvature. At right angles to *OA* and *OB*, draw the tangents *AE* and *BE*, and join *OE* by a straight line cutting the arc at the crown *D*. Of course, *DE* will be the length $R(\sec \frac{\theta}{2} - 1)$, and the axial stress P + Q, multiplied by *DE*, will give us the bending moment at *D*. Then, through the three points, *A*, *E*, and *B*, draw the circular arc *AEB*, the circle passing also through *O*, and between the two arcs draw any number of polar ordinates, such as *gk* normal to the arc *ADB*. Then at any point, *g*, the bending moment, will be proportional to *gk*, and equal to that length multiplied by P + Q.

The diagram may be drawn in another form as follows:—

Considering the forces P + Q and S as external forces, their resultant H will have the direction of the chord-line *AB*, and the magnitude $H = (P + Q) \sec \frac{\theta}{2}$. It follows that if we simply draw the chord *AB*, the moment at any point *g* in the arc will be proportional to the perpendicular *gm*, and will be equal to that ordinate multiplied by H, or by $(P + Q) \sec \frac{\theta}{2}$.

EXAMPLE.—In the particular case illustrated in Art. 11, the angle θ subtended by the bend *ADB* was 45°, and $\sec \frac{\theta}{2}$ would therefore be equal to 1·08248; so that with the given radius of 20 feet, the quantity $R(\sec \frac{\theta}{2} - 1)$ would be—

$$20 \times 0{\cdot}08248 = 1{\cdot}6476 \text{ feet}$$

Then, if the external diameter d_1 of the sliding neck is 39 inches, as assumed in that example, the gross axial stress P + Q, under a head of 320 feet, will be 73·94 tons, and the bending moment at the centre of the arc will be—

$$M_D = 73{\cdot}94 \times 1{\cdot}6476 = 125{\cdot}82 \text{ foot-tons}$$

Art. 15. The Pipe-bend acting as a Cantilever.—Let the horizontal pipe *AB*, in Fig. 22, be connected with the vertical pipe *DE* by a bend *BCD* turning through an arc of 90°, and let the pipe be fitted with an expansion joint at B. This case, or something very much like it, is of frequent occurrence in the steam pipes of large marine engines, the pipe *DE* being very often attached to the side of the valve chest by a flanged joint at *E*.

We may regard the bent pipe *BCDE* with its fluid contents as forming together one body, which as a whole is subjected to the external forces H and V. The former of these is of course the gross axial stress P + Q, or $pd_1^2\frac{\pi}{4}$, taking effect at the sliding joint, whose bore is d_1, while the vertical force V will really be determined by the area of the orifice in the side of the valve chest. This will generally be a circular orifice, whose diameter *d* is the same as that of the pipe, and then, of course, $V = P = pd^2\frac{\pi}{4}$.

The force H must evidently be resisted by the bent pipe acting as a cantilever, fixed at the base *E*; and at the same time the flange joint at *E* will in all probability have to resist the force V in direct tension, because the straight pipe *AB* will hardly afford any material resistance at *B* to the action of this force.

At the base E_1E_2 of the cantilever we shall have, therefore: (1) a bending moment $M_e = HX_e$, in which X_e denotes the height measured from the base *E* to the horizontal line of action of the force H, or to the centre of the expansion joint; (2) a shearing force parallel to H and having the same magnitude; (3) a direct tensile stress due to the vertical force V, which may be taken as uniformly distributed over the horizontal cross-section of the pipe.

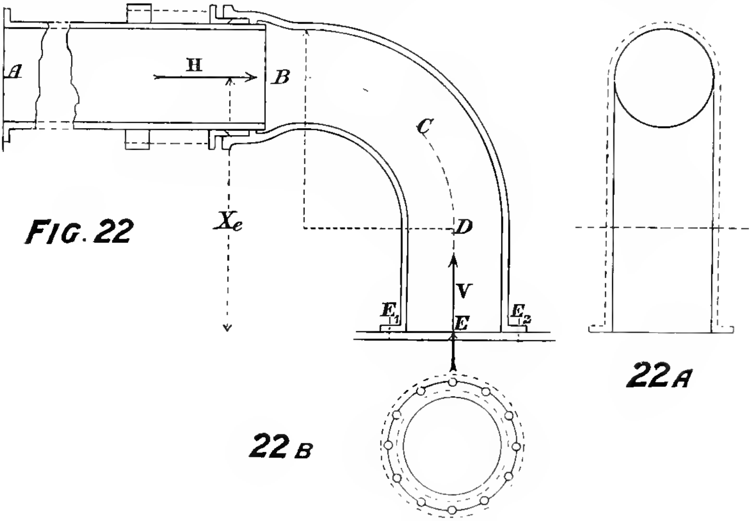


FIG. 22

22A

22B

The bending moment M_e will be resisted by vertical longitudinal stresses, which at E_1 and E_2 will have to be algebraically added to the vertical stress due to the force V.

Instead of referring to the axial fluid stress, or pressure of fluid upon fluid, we might arrive at the same results by considering the pressure of the fluid on the internal surfaces of the bend. This unbalanced pressure would produce, of course, a displacing force acting normally to the curve of the bend, and uniformly distributed around the arc; but the horizontal and the vertical components of the unbalanced pressures are easily measured by the method already described, and the projections of Figs. 22*a* and 22*b* show at once the magnitude and the centre of action of the horizontal and vertical forces.

EXAMPLE.—Suppose the pipe illustrated in Fig. 22 to have an internal diameter of 18 inches and a thickness of half an inch, the bore of the expansion joint being 19 inches. Also let the height of the cantilever X_e be 45 inches (measured from base to centre of expansion joint).

Then, if the pressure p is 70 lbs. above atmospheric pressure, the horizontal thrust H and the shearing force S at the flange joint will amount to—

$$H = S = 70 \times 19^2 \times 0{\cdot}7854 \div 2240 = 8{\cdot}86 \text{ tons}$$

the bending moment at the base of the cantilever will be—

$$M_E = 8{\cdot}86 \times 45 \text{ inches} = 398{\cdot}7 \text{ inch-tons}$$

while the vertical force resisted by the flange joint will be—

$$V = 70 \times 18^2 \times 0{\cdot}7854 \div 2240 = 7{\cdot}95 \text{ tons}$$

To trace the further effects of the axial fluid stress upon the engine framing, it would of course be necessary to define the conditions in each case; but it is obvious that the thrust H, delivered upon the end of the cantilever, would commonly produce in the engine framing certain racking or twisting or bending stresses, which could generally be calculated if all the conditions were known.

The manner in which the reaction of the fluid stress would be felt by the straight pipe *AB* will form the subject of the next chapter.

Art. 16. Guard Bolts.—It has occasionally (and not very infrequently) happened that steam-pipes, constructed in the manner above described, have given way under the action of the displacing forces; and it is probably owing to these "accidents" that engineers have adopted the plan of providing "guard bolts" for the expansion joints of large steam-pipes. For, whether these forces take effect by displacing the bend or by displacing the pipe *AB*, their tendency is always to draw out the pipe from the expansion joint; and the guard bolts may at least serve as a check to prevent the total withdrawal of the pipe in the event of any yielding of the structure under the displacing forces.

When, in such an event, the guard bolts come home against their bearings, they may transmit a portion or the whole of the axial fluid stress by direct longitudinal tension through the metal of the pipe; but it must be remarked that the mere employment of these guard bolts does not dispense with the necessity of making the structure strong enough at every point to resist the displacing forces, and, indeed, strong enough to resist them without any aid from the guard bolts; for, so long as the expansion joint is really free to slide as an expansion joint, it is obviously impossible that the guard bolts can come into action at all; and, in the case of a high-pressure steam-pipe, it is evident that the greatest fluid pressure and the greatest displacing force occur at the time when the temperature is highest and when the guard bolts are pushed farthest from their bearings by the expansion due to that high temperature.

Art. 17. The Transverse Strength of Cylindrical Pipes.—If a pipe is made of the same thickness all round, the neutral axis will of course be a diametral line at right angles to the plane of flexure; and the direct tensile or compressive stress in any longitudinal fibre of the metal pipe will have an intensity proportional to the perpendicular distance of that fibre from the neutral axis, on one side or the other. Thus, the fibre stress due to any bending of the pipe will have a varying intensity in different parts of the cross-section, and will reach its maximum at the extreme edges of the section farthest from the neutral axis.

To find the maximum intensity of fibre stress (whether in tension or in compression) due to any given bending moment, we may first calculate the moment of inertia by the well-known formula—

$$\mathrm{I} = \frac{\pi}{64}(d_1^4 - d^4) \quad . \quad . \quad . \quad (14)$$

in which d is the internal, and d_1 the external, diameter.[1]

Then the maximum intensity of stress f_1 in the extreme fibre, due to any bending moment M, will be exactly expressed by—

$$f_1 = \frac{\mathrm{M}d_1}{2\mathrm{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

When, however, the thickness t is small in comparison with the diameter, we can find f_1 by an approximate formula which is more easily worked out, and is often sufficiently accurate. For if D represents the mean diameter $d + t$ of a thin pipe, the

---

[1] It may be useful to note in passing that the sectional area of the same pipe will be given by—

$$\mathrm{A} = \frac{\pi}{4}(d_1^2 - d^2) \quad . \quad . \quad . \quad . \quad (14a)$$

and the radius of gyration will have for the value of its square—

$$r^2 = \frac{\mathrm{I}}{\mathrm{A}} = \frac{d_1^2 + d^2}{16} \quad . \quad . \quad . \quad . \quad (14b)$$

moment of resistance will be very nearly equal to the sectional area A, or πDt multiplied by $\frac{Df_1}{4}$, so that approximately—

$$f_1 = \frac{4M}{AD} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Example.—At the base of the cantilever illustrated in Art. 15, it was found that the bending moment HX_e, due to a steam-pressure of 70 lbs. per square inch, would amount to 398·7 inch-tons.

The internal diameter of this pipe being 18 inches, and the thickness half an inch, the mean diameter D will be 18·5 inches, and the sectional area will be—

$$A = 18{\cdot}5 \times \pi \times 0{\cdot}5 = 29{\cdot}05 \text{ square inches}$$

The maximum fibre stress due to this bending moment will therefore be approximately—

$$f_1 = \frac{4 \times 398{\cdot}7}{29{\cdot}05 \times 18{\cdot}5} = 2{\cdot}97 \text{ tons per square inch}$$

this being the tensile stress on the side E_1 and also the compressive stress on the side E_2 in Fig. 22.

If, however, we make the calculation by means of formula (14), we shall find that the true value of f_1 is a little greater, and amounts to exactly 3·04 tons per square inch at the extreme outer skin of the pipe.

In the example here chosen, it was shown, in Art. 15, that in addition to the bending stress the pipe would be subjected to the direct force V, which would be resisted by a tensile stress uniformly distributed over the whole section E_1E_2, whose area is 29·05 square inches. The uniform tensile stress due to the force V will therefore be—

$$7{\cdot}95 \text{ tons} \div 29{\cdot}05 = 0{\cdot}27 \text{ ton per square inch}$$

In this case, therefore, the tensile stress at E_1 will be—

$$3{\cdot}04 + 0{\cdot}27 = 3{\cdot}31 \text{ tons per square inch}$$

and at E_2 we shall have a compressive stress amounting to—

$$3{\cdot}04 - 0{\cdot}27 = 2{\cdot}77 \text{ tons per square inch}$$

The conjugate stresses at E_1 will then consist of the hoop tension $f = \frac{70 \times 18}{2 \times \frac{1}{2} \times 2240} = 0{\cdot}55$ ton per square inch, and the longitudinal tension f_1, which is about six times as great as the hoop tension.

Art. 18. The Transverse Strength of Pipe Joints.—When a line of jointed pipe is subjected to any bending forces, the longitudinal stresses or fibre stresses due to the bending moment at any joint must be transmitted through the joint itself.

Thus, if we take first the case of a line of flanged pipes united by ordinary flange bolts, and neglect in the first instance any question of initial stress, it appears certainly that while the compressive stresses on one side of the neutral axis are transmitted by the pressure of the flanges, the tensile stresses on the other side must be transmitted through the bolts, and that the stress in each bolt will be nearly proportional to its distance from the neutral axis.

The bolts will, in most cases, be spaced at uniform intervals around the "pitch-circle," whose diameter we may denote by D_1, and, on the assumption above mentioned, we may take it that the transverse strength of the bolts will be nearly the same as though they were replaced by a continuous circle of metal, or thin pipe, having the diameter D_1 and a sectional area equal to the combined areas of all the bolts in the circle. The maximum stress per square inch in the outermost bolt would then be given by the same formula $f_1 = \frac{4M}{AD_1}$, in which A now represents the combined sectional area of all the bolts, and D_1 the diameter of the pitch-circle.

EXAMPLE.—Suppose that the steam-pipe already illustrated in Fig. 22 is united to the valve-chest at the base of the cantilever by a flange joint, as sketched in Fig. 22*b*, the flange bolts, 12 in number, being uniformly spaced around the pitch-circle, which is 21 inches diameter. Then, if each bolt has a diameter of $1\frac{1}{4}$ inch inside the thread, its net sectional area will be about 1·23 square inches, and the combined section of the 12 bolts will have an area of 14·76 square inches.

The bending moment of 398·7 inch-tons, above calculated, would then produce in the outermost bolt a tensile stress which may be approximately estimated at—

$$f_1 = \frac{4 \times 398{\cdot}7}{14{\cdot}76 \times 21} = 5{\cdot}14 \text{ tons per square inch}$$

As the flange joint will also have to resist the direct stress due to the force V of 7·95 tons, the additional stress of $\frac{7{\cdot}95}{14{\cdot}76} = 0{\cdot}50$ ton per square inch would bring up the total stress to 5·64 tons per square inch in the outermost bolt, without making any allowance for the initial stress that may be imposed upon the bolts. In the practical process of "making" the joint, the flange bolts will no doubt be so firmly screwed up as to throw them all into considerable tension, and to bring the flanges together with a certain compressive force. Whatever compressive force may be requisite for this purpose in a joint otherwise unstrained, will probably also be necessary when the bolts are utilized to transmit such structural stresses as those above considered; and it would therefore appear that the initial stress (required for tightness) should be added to the structural stresses.

If we go on to consider next the case of a riveted pipe of steel plate in which the lengths are united, as in ordinary boiler work, by a lap joint, single or double riveted, it will be seen that in all such cases the longitudinal stresses due to any bending moment, or to any direct longitudinal forces, will have to be resisted by the shearing strength of the rivets. And we may again assume that the shearing stress borne by the individual rivets around the circumferential joint will be proportional to the direct stress, at each point in the circle, taking effect in the metal of the pipe itself—the stress being zero at the neutral axis and reaching a maximum at the two edges of the circle farthest from the neutral axis.

Thus, for any bending moment M we may find the maximum shearing stress by the formula already employed—*i.e.* if A denotes the whole rivet section around the circumferential joint, and D_1 the diameter at the joint, the maximum shearing stress will be—

$$s_1 = \frac{4M}{AD_1}, \text{ very nearly} \quad . \quad . \quad . \quad (17)$$

It is unnecessary to remark that in such riveted joints the net plate section would have to be proportioned to the longitudinal stress as well as the rivet section.

CHAPTER IV.

THE BUCKLING TENDENCY IN STRAIGHT PIPES UNDER UNIFORM FLUID PRESSURE.

Art. 19. Axial Stress in the Straight Fluid Column.—In the case of a long straight pipe supported only at the ends, or supported between those points in such a way as to permit of free lateral flexure, it will sometimes be necessary to consider the pipe as a *fluid column* liable to buckling under the axial fluid stress.

The conditions under which it would be possible for a pressure-pipe to be actually buckled in this way are possibly of somewhat rare occurrence, but the tendency will certainly be present under conditions that are sometimes met with in actual engineering practice, as, for example, in the straight steam-pipe AB illustrated in Art. 15.

The buckling of a slender column of solid elastic material, under the action of an end-thrust, is a familiar fact, depending upon known mechanical principles, and the same principles must have an application to the cases which we now have to consider. It might, indeed, be sufficient to treat the question by direct reference to the longitudinal compressive forces or stresses; but in this problem, as in previous ones, it will probably be better to trace the buckling action of the fluid stress to its distinct proximate cause, viz. the unbalanced pressure of the fluid against the internal surface of the pipe.

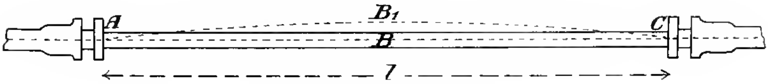
In some cases the metal pipe may itself be subjected to a direct compressive stress Q due to the fluid pressure upon the annular end of the pipe, as in previous examples, and it is easy to see that this end-thrust would be accompanied by a certain buckling tendency, just as in any other hollow cylindrical column; but it is not quite so plainly apparent that the fluid stress P will act in a similar manner, and that the pipe may be buckled without undergoing any compressive longitudinal stress, and solely by the action of internal pressure.

Art. 20. Displacing Forces due to Elastic Deflection. — To illustrate the manner in which a pipe, initially straight, may be subjected to the action of displacing forces, let us suppose the pipe *ABC* in Fig. 23 to be carried at each end in expansion joints, which are themselves firmly fixed to a substructure, while between them the pipe is free to deflect in any transverse direction, the ends being regarded as merely *supported* by the glands, and not rigidly fixed in direction.

We may further suppose that the weight of the charged pipe is counterbalanced, and then it must be admitted that the internal fluid pressures in the straight cylindrical pipe will exactly balance each other, and can produce *no* displacing force whatever *so long as the pipe is straight.* But a precisely similar statement might be made in regard to the analogous case of the solid column, in which the load can produce no bending moment so long as the column is straight; and when the fact is acknowledged, in either case, the real question at issue is not disposed of. In both cases there may be no recognisable force by which we can account for the beginnings of deflection; but in both cases the requisite force is forthcoming as soon as deflection commences. In the solid column, the deflection gives rise at once to a certain bending moment; and in the pipe the deflection gives rise at once to a certain displacing force consequent upon the curvature of the pipe.

To ascertain the stability of the pipe, therefore, we shall have to pursue the same inquiry as in the case of a slender column— *i.e.* assuming hypothetically a certain slight curvature, we must calculate the displacing force that would arise simultaneously with that curvature, and see whether it is greater or less than the resisting forces that would be called into play at the same instant of time.

FIG. 23



Assuming, then, that the pipe *ABC* is, or may be, bent elastically out of the straight line to a slight certain curvature AB_1C, or slight deflection $\delta = BB_1$, it has already been found in Art. 9 that if R is the radius of curvature, the displacing force will be a distributed radial pressure whose value per foot lineal is $n = \frac{P}{R}$; and as δ will be taken to be very small in comparison with the span l, the pressure may be called a lateral or transverse pressure, acting upon the pipe just as a distributed load acts upon a beam that is supported at each end. This brings the problem at once within sight of an easy approximate solution.

Art. 21. The Conditions of Equilibrium broadly considered.—Without stopping to inquire how or by what means the straight pipe could ever become bent by internal pressure, let us assume that it *has* acquired a slight curvature (within the elastic limit), and let us ask the more important question whether the deflection will now increase, or whether the pipe will straighten itself by its own elastic reaction or resilience. Evidently the question will depend upon the relative magnitude of the forces which are here opposed to each other—the displacing force, tending to push on the deflection, and the elastic resilience of the pipe, tending to resist any further deflection, and to restore the pipe, if it can, to the straight form.

We may take first the displacing force, and to make this calculation as simple as possible, let us assume, in the first instance, that the curve AB_1C in Fig. 23 is a very flat segment of a circle. Then the radius of that circle will be very nearly $R = \frac{l^2}{8\delta}$, and the displacing force will be equivalent to a uniformly distributed load (or lateral pressure) whose value per foot lineal will be $n = \frac{P}{R} = P\frac{8\delta}{l^2}$, nearly.

With this uniform load upon a beam supported at each end, the bending moment at the centre of the span can at once be calculated by the well-known rule, and we have for that bending moment $M_B = \frac{nl^2}{8} = P\delta$.

We may next turn to the elastic reaction or resilience of the pipe, which is the only force concerned in resisting the displacing force. The elastic pipe will be a beam of uniform section, and if I denotes the moment of inertia of its section, we know very well that it will take a certain deflection—no more and no less—under a uniformly distributed load of any given intensity n_1; and by the well-known rule that deflection will be—

$$\delta = \frac{5}{384} \cdot \frac{n_1 l^4}{EI}$$

the deflection δ being proportional to the load n_1 so long as the stress is kept within the elastic limit. But, putting this relationship in another form, let us say that when the beam bends to a certain deflection δ, it will carry a certain distributed load—no more and no less—and evidently the load that it *will* carry with that particular deflection is simply—

$$n_1 = \frac{384}{5} \cdot \delta\frac{EI}{l^4}$$

Then taking this quantity n_1 as representing, for our present purposes, what may be called the distributed reactionary force of the elastic beam, we have only to compare its magnitude with the magnitude of the displacing force n, against which it is opposed, and which is also a uniformly distributed force. This comparison will show us at once whether the pipe will buckle or whether it will straighten itself.

Putting the two formulæ for n and for n_1 side by side, we see that the displacing force n will be greater than the opposing resistance n_1, when $P\frac{8\delta}{l^2}$ is greater than $\frac{384}{5}\delta\frac{EI}{l^4}$, or when P is greater than $\frac{48}{5} \cdot \frac{EI}{l^2}$.

Apparently, then, the question at issue does not depend upon the value that may be arbitrarily selected for the deflection δ in making this comparison. As the deflection increases the elastic reaction of the pipe increases uniformly, but so also does the displacing force, and the question depends only on the relative *rates* at which these forces increase. If, then, we find that with a deflection of one inch the displacing force would overpower the elastic resistance, we may conclude that the same thing would be true if the deflection were only a quarter of an inch or only the thousandth part of an inch, and we need not go any further to discover a reason why the pipe should begin to curve from a condition of initial straightness. The critical value of the axial fluid stress P, that will determine the question of buckling or not buckling, may be denoted by P_c, and we may write $P_c = \frac{48}{5} \cdot \frac{EI}{l^2}$. If P is greater than P_c, the pipe will buckle; but if P is less than P_c, the pipe will not buckle; and if we forcibly drag it out of the straight, it will spring back again as soon as we let it go.

Art. 22. Exact Determination of the Critical or Buckling Pressure.—The reasoning put forward in the last article is not quite correct, for we assumed that the unknown curve of the beam would be a curve of segmental form. We had no right to assume, as we have done, that the radius of curvature would be uniform throughout, and if it is not uniform the displacing force will not be uniformly distributed.

Without making any tentative guesses as to the true form of the curve, we may set in order the known relationships that are concerned in this question, and its solution will then become manifest.

1. With *any* given distribution of load, represented, let us say, by the ordinates of a load diagram, we can construct the corresponding curve of moments, and it is certain that this curve can have only one form for the given distribution of load.

2. From this diagram of moments, we can, by a similar process and with equal certainty, construct the deflection curve for a homogeneous beam of uniform section.

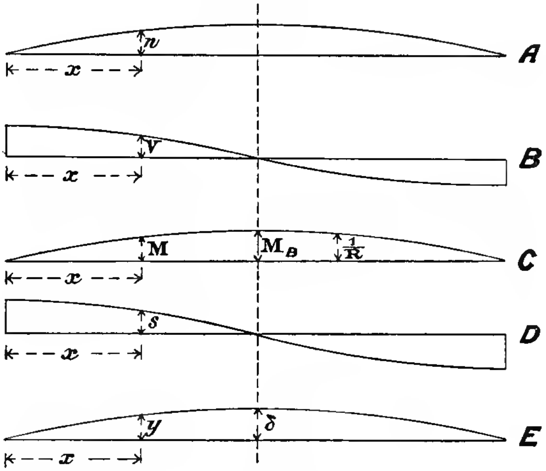
3. In the deflection curve thus constructed we can find the slope at any given point, and also the radius of curvature R, or the quantity $\frac{1}{R}$, which is often called the "curvature" of the beam; and it is also known that this quantity $\frac{1}{R}$ will be at all points proportional to the bending moment.

4. It follows, of course, that if this varying quantity $\frac{1}{R}$ is represented by the ordinates of a diagram, that diagram must be bounded by a curve similar to the curve of moments; and it also follows that the curve must be similar to the curve of the load diagram from which the whole series of diagrams has to be constructed, because the load or displacing force is proportional to $\frac{1}{R}$, and is in fact $n = \frac{P}{R}$.

There is nothing to fix the form of any one of the curves *à priori*, but this last condition furnishes the required solution, and fixes the form of them all. The actual distribution of the displacing force *depends* upon the curve of the pipe, while it *governs* the form of the diagram of moments and the form of the deflection curve. But the deflection curve *is* the curve of the pipe.

Putting the several curves one below the other, in the order of cause and effect as we usually conceive that relationship, let the diagram *A* in Fig. 24 represent by its ordinates the unknown intensity of the load or displacing force *n*; while *B* shall represent the shearing forces, and *C* the bending moments resulting from this distribution of the load. Then, to follow out the elastic deformations consequent upon these bending stresses, let *D* represent by its ordinates the slope of the bent beam, and *E* the curve of deflection. It is known that *B* can always be obtained from *A* by integration, or from *C* by differentiation; and it has been shown elsewhere[1] by the author that just as

FIG. 24



B and *C* are usually constructed from *A*, so *D* and *E* can be constructed from *C*. Indeed, it will easily be seen that the slope *s*, whose varying values are shown in diagram *D*, is simply the differential of the deflection ordinate *y* in diagram *E*; and that the quantity $\frac{1}{R}$ will be proportional to the differential of *s*, if the deflection is small; while a well-known theorem informs us that the same quantity is proportional also to the bending moment M represented by the ordinates of diagram *C*.

All these relationships are generally to be traced in any elastic beam of uniform stiffness; but, in the case we are now considering, the special condition which determines the form of all the curves, is that the curve *A* must of necessity be similar to the curve *C*, because $n = \frac{P}{R}$.

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[1] *Vide* Paper on "Continuous Girders," in the *Transactions of the Inst. C.E.*, vol. lxxiv.; also "Bridge Construction," chap. viii. p. 113.

Hence, it follows that, in the deflection curve *E*, in which y is some unknown function of x, that function must be such that its second and its fourth differential shall have the same form, and this condition is satisfied by the curve of sines.

Thus, in diagram *E*, let δ represent a small deflection of the pipe at the centre of its length, and for all values of x between zero and π (or 180°) let the corresponding ordinates be $y = \delta \,.\, \sin x$; then the curve so constructed may be taken to represent either of the curves *A*, *C*, or *E*, so that, by using a suitable vertical scale in each case, it will measure for us either the deflection ordinate y, the curvature $\frac{1}{R}$, the bending moment M, or the intensity of the displacing force n. In the same way, it will be seen that the curve of cosines will give us the vertical shearing force in *B*, or the slope of the pipe in *D*.

It is now obvious that the distribution of the displacing force is *not* uniform, as we assumed it to be in the last article; and that the curve of the bent pipe is not exactly a flat segment. When the deflection is small there is but little difference between the two curves; but taking now the curve of sines, which satisfies all the mechanical conditions, we shall have to modify slightly the results that were obtained by using the other; and we find for the critical value of the axial fluid stress—

$$P_c = \pi^2 \frac{EI}{l^2} = 9{\cdot}87 \frac{EI}{l^2}$$

This expression will be recognized at once[1] as being identical with a well-known formula for the "breaking load" or "buckling load" of a slender column of solid material; and, so far as the buckling tendency is concerned, it is evident that the axial fluid stress P in a straight pipe is equivalent to so much end-thrust, although the fluid pressure takes effect, not upon the end of the pipe, but upon its internal surface.

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[1] *Vide* Paper on the "Practical Strength of Columns," *Trans. Inst. C.E.*, vol. lxxxvi.; also "Bridge Construction," chap. x. p. 153.

The critical fluid pressure per square inch may be denoted by p_c, and if the metal of the pipe is absolutely free from longitudinal compressive stress, its value will be—

$$p_c = \frac{4P_c}{\pi d^2} = \frac{4\pi EI}{l^2 d^2}$$

Art. 23. Effective Length for Flexure and Effective Axial Thrust.—The length l, in the formulæ above given, is, of course, the length of the span *AC* between supports, or the length of pipe that is free to deflect in any one direction, just as in the column formula it represents the length of a column with rounded or hinged ends. When the pipe is carried between two expansion joints, as in Fig. 23, we can hardly attribute to those joints any rigid fixity of direction; and the length l would, therefore, be measured from joint to joint.

But in other cases, as perhaps in the steam-pipe illustrated in Fig. 22, the pipe may be rigidly fixed at one end *A*, while the other is carried in an expansion joint, which is itself fixed in position, although affording no fixity of direction; and in such a case the effective length l for flexure might probably be estimated at two-thirds to three-quarters of the total length.

In this, as in so many other respects, the pipe may be considered simply as a column subjected to a certain load, or exterior end-thrust.

The effective value of this axial thrust must next be considered, and a little reflection will show that, in all cases, it must be taken to be the algebraical sum of the longitudinal stresses P and Q in the fluid and in the solid portions of the column, as already indicated in Arts. 11 and 12.

The axial fluid stress P is equivalent to so much end-load, and, if the metal of the pipe is also subjected to the end-thrust Q, due to fluid pressure upon its annular end, the total axial thrust will be $P + Q = pd_1^2\frac{\pi}{4}$, in which d_1 is the diameter of the bore of the gland.

To avoid buckling in this case, the effective thrust P + Q must, of course, be less than the critical value P_c as determined in Art. 22.

The external load being thus determined by the bore d_1, that load will, of course, be transmitted through the whole length *ABC*, and will represent the sum of the longitudinal stresses

P and Q at every cross-section between *A* and *C* in Fig. 23. If we make the intervening length of pipe with an internal diameter *d* equal to the bore of the gland, the fluid axial stress at any intervening cross-section will be equal to the total external thrust $pd_1^2\frac{\pi}{4}$; and, therefore, the metal pipe will be relieved of all longitudinal compressive stress throughout the free length intervening between the expansion joints. But notwithstanding its entire freedom from any longitudinal compression, the pipe will be buckled by fluid pressure alone if the effective axial stress P is greater than the critical value P_c.

It would, of course, be possible to go a step further, and by making the intervening length of pipe with an internal diameter *greater* than the bore d_1, the pipe would be thrown into longitudinal *tension* throughout its free length; but would, nevertheless, be buckled if the algebraical sum P + Q, or the end-thrust $pd_1^2\frac{\pi}{4}$ (which is the same thing), were greater than the quantity P_c.

In short, it is evident that the buckling of such a pipe would not depend upon the transmission of compressive stress through the *metal* of the pipe, but upon the compressive stress transmitted through *the column as a whole,* including the fluid and the solid portions of its cross-section; and obviously, if the two ends of a straight pipe are closed by covers, so that the pressure P upon each cover is transmitted in direct tension $Q = -P$ throughout the pipe, the algebraical sum P + Q will be zero, and there will be no buckling tendency, however flexible the pipe may be.

Here, however, it may be as well to remark that if such a tubular column with closed ends were subjected to an external load W, that load would at once become the measure of the buckling tendency, and also of the effective axial thrust P + Q. In such a case the longitudinal stress Q in the metal of the pipe might be either positive or negative or zero, but the buckling tendency would in either case be that due to the load W.

Whatever the external conditions may be, the fluid stress P will be in operation in all cases, but the longitudinal tensile stress Q, in the metal of the pipe will depend upon those conditions very largely. As already mentioned, it will often be difficult to ascertain the longitudinal stress Q, and it may then be difficult to say whether there is any buckling tendency; but in other cases the conditions are more clearly defined, and when an expansion joint is introduced into the pipe the conditions are quite definite, and there can be no doubt that the buckling tendency will then have the values above found.

It would hardly be justifiable, however, to employ a "column formula" of the usual kind in determining the requisite strength of such pipes, for the pipe differs from the column in one important respect. The column whose breaking weight is given by the "column formula," or by recorded experiments, is subjected to direct compressive stress as well as to the liability of buckling, while the pipe suffers no direct compressive stress, or, at all events, none worth speaking of.

If, then, the section of the pipe is such that the quantity $EI\frac{\pi^2}{l^2}$ is greater than the axial stress $P + Q$ by a reasonable margin of safety, we may reasonably conclude that the pipe will not begin to bend; and if it does not begin to bend, the buckling tendency will exist only as a purely inoperative tendency, producing no stress and no actual effect of any kind upon the pipe.

CHAPTER V.

FLUID ARCHES.

Art. 24. Pressure Pipes used as Tubular Arches.—In extensive works of water supply, gravitation mains are often carried across rivers or valleys by bridges constructed for that purpose; and in several cases it has been proposed to construct such bridges in the form of a tubular arch, in which the pipes themselves constitute the hollow arched ribs. This principle has been followed in a few structures, designed for the conveyance of either water or gas, and may perhaps be found useful in other situations.

Such an arch must evidently be regarded as an upright bend, like the segmental bend of Art. 10, in which the "displacing force" acts as a distributed pressure in a radial (upward) direction, and in opposition to the load or weight of the structure; and the load being thus in some measure supported by the fluid stress, the compressive stress in the tubular arch would be correspondingly reduced. If the displacing force were great enough, it might evidently relieve the metal arch of *all* its usual compressive stress, so that the load would be carried wholly by the arch of water, and not at all by the arch of steel; and if the displacing force were still greater, it would tend to lift the whole superstructure, or if the tubes were anchored down to the abutments would throw the metal arch into tension, like the chain of a suspension bridge inverted.

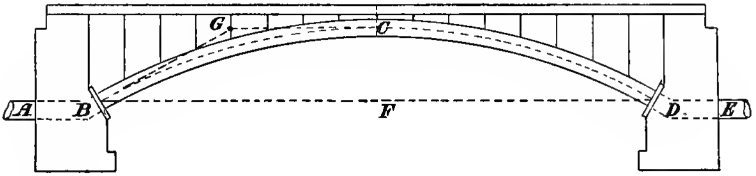
The structure considered simply as an arch, having a given figure and carrying a given load, will, of course, be subject to the well-known principles by which the axial thrust at any point may be calculated, and by which the bending stresses due to deformation under the load, or due to change of temperature, may also be approximately determined in arched ribs of known section; but, on the other hand, we have to bring into the calculation, in addition to all these, the displacing forces which are due to fluid reaction, and which will constitute the special feature of such structures.

It will hardly be necessary to pursue this question into all its details, and it will be sufficient to take a few simple cases which will illustrate the effects of the fluid stress in tubular arches charged with hydraulic or pneumatic pressure.

Art. 25. Direct Longitudinal Stress in Charged Tubular Arches.—Let us suppose the bridge sketched in Fig. 25 to consist of two (or more) arched ribs of hollow cylindrical section, placed side by side and united by suitable cross-bracing; and let each of these arched ribs form part of a continuous line of pressure main, the ribs being also abutted upon the skew-backs at *B* and *D* by exterior flanges, which take a bearing upon suitably formed annular bedplates.

For the sake of simplicity, we shall suppose the load to consist wholly of dead load, including, of course, the weight of the arched pipes and their fluid contents, and that the figure of the arch is designedly adapted to the known distribution of the load, so that the line of pressure shall coincide with the neutral axis of the rib or the curved centre-line of the arched pipe. And we may perhaps also assume that the fluid pressure is uniform throughout the pipe, as it would be practically if the fluid were gas or compressed air, and as it would be very nearly if the pipe were charged under a great hydrostatic head, such as would be likely to occur at the valley crossings of a gravitation main.

FIG. 25



Then, if the internal diameter d of the arched pipe is first assumed to be uniform throughout the arch, the axial fluid stress will at all points have the same value, $P = pd^2\frac{\pi}{4}$; while the arch thrust will, of course, differ slightly at different points in the curve.

Thus, if we take first the case of a *uniform* load—*i.e.* a load of uniform weight w per foot of span, supported by a *parabolic* arch—we may denote the total load by W, the span (*BD*) by L, and the rise of the arch (*CF*) by D. The horizontal thrust at the crown of the arch will then be given by the familiar formula—

$$H = W\frac{L}{8D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the direct axial thrust at any point in the rib will be—

$$C = H \sec \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ϕ is the varying inclination of the rib.

The horizontal distance *CG* to the intersection of the tangent *BG* will, in this example, be equal to one-fourth of the span, and the point *G* will, of course, coincide with the centre of gravity of the uniform load upon the half-span.

But, in most cases, the centre of gravity will be somewhat nearer to the abutment, and consequently the curve of pressure will be more nearly segmental than parabolic. In the equilibrated segmental arch the centre of gravity of the half-load must coincide with the intersection *G* of the two equal tangents, and the true value of the horizontal thrust would then be—

$$H = \frac{W}{2} \cdot \cot \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (3)$$

in which θ is the angle subtended by the whole arc *BCD*, and $\frac{\theta}{2}$ is the angular inclination of the tangent *BG* to the horizontal. In either form of rib, however, we may express the direct thrust by $C = H \sec \phi$; and we have already seen that this thrust must represent the algebraical sum of the compressive axial stresses in the fluid and in the solid portions of the arch. The direct longitudinal stress in the metal of the tube will therefore be—

$$Q = C - P = H \sec \phi - P \quad . \quad . \quad . \quad . \quad (4)$$

If Q is positive the tube will be in compression, but if it is a negative quantity it will mean that the tube is in tension like the chain of a suspension bridge.

Example 1.—Let us suppose the bridge sketched in Fig. 25 to consist of a pair of tubular arched ribs of riveted steel plate, the internal diameter of each tube being 36 inches throughout the span, and the same as the internal diameter of the horizontal water pipes *AB* and *DE* of the gravitation main, in which the arched tube forms a part of the continuous line of pipe. Let the span (*BD*) be 120 feet, and the rise 12 feet, or one-tenth of the span, and suppose, first, that the arch is of parabolic form, and the load uniformly distributed.

The actual weight of such pipes, made of half-inch steel plate with double butt joints, but without any stiffeners, may be estimated at 2 cwt. per foot lineal, while the contained water in each pipe would weigh 4 cwt. per foot, so that the total weight of the two pipes would be 12 cwt. per foot lineal; but we will here suppose that the total dead weight of the structure as sketched, including roadway girders, spandrils, and bracing, amounts to 20 cwt. per foot of span, or W = 120 tons, and we will examine the equilibrium of the bridge under this uniform load of 1 ton per foot when the pipes are charged under a hydrostatic head of 320 feet.

Taking a cross-section through the arch at the crown, we have the horizontal thrust due to the load W, or—

$$\text{H} = 120 \times \frac{120}{8 \times 12} = 150 \text{ tons}$$

But the axial fluid stress in each pipe under a head of 320 feet will be 63 tons (as already calculated in Art. 9), so that a fluid stress of 126 tons will act in direct opposition to the arch thrust at the crown; and it follows that the compressive stress in the tubes themselves can only amount to—

$$\text{Q} = \text{C} - \text{P} = 150 - 126 = 24 \text{ tons}$$

or 12 tons in each tube.

At the springing of the parabolic rib, we should have—

$$\sec \phi = \frac{\sqrt{30^2 + 12^2}}{30} = 1{\cdot}077$$

and here the arch thrust would be increased to C = 150 × 1·077 = 161·5 tons; and the compressive stress in the metal tube, or the thrust delivered upon the annular bedplate, would be about—

$$\text{Q}_\text{B} = 161{\cdot}5 - 126{\cdot}0 = 35{\cdot}5 \text{ tons}$$

To be a little more accurate, we should perhaps say that if the head is exactly 320 feet at *C* it will be 332 feet at *B*, and the axial fluid stress being thus increased to 130·7 tons, the thrust of the tubes would be—

$$\text{Q}_\text{B} = 161{\cdot}5 - 130{\cdot}7 = 30{\cdot}8 \text{ tons}$$

or 15·4 tons in each tube.

Example 2.—If, in the same bridge, we suppose the pipes to be charged under a hydrostatic head of 381 feet (measured at the crown), it is easy to calculate that the fluid stress at the crown would be P = 150 tons, very nearly; and the tubes would then be relieved of all longitudinal stress at the crown; while at the springing we should have a hydrostatic head of 393 feet, and as P would then amount to 154·8 tons, there would be at the springing a slight compressive stress—

$$\text{Q} = 161{\cdot}5 - 154{\cdot}8 = 6{\cdot}7 \text{ tons}$$

Or, again, if we suppose the hydrostatic head at B to be 402 feet, the axial fluid stress at that point would be nearly 161·5 tons. In this case there would be no compressive stress in the tubes at B, and they would exert no thrust upon the abutment. The thrust of the fluid arch would, of course, be delivered upon the fluid in the buried main AB, the horizontal component of that thrust being exactly met and balanced by the fluid axial stress in the horizontal main, while its vertical component would be just $\frac{W}{2}$, and would be carried as a vertical force or load upon the abutment wall. But, at the crown of the arch, we should now find a slight *tensile* stress; for although the head would be reduced at that point to 390 feet, yet the fluid axial stress would amount to 156·7 tons; and as the algebraical sum of the stresses in tube and in fluid must be 150 tons, the tubes must be under a tensile stress of 6·7 tons.

Lastly, if the hydrostatic head were greater than 402 feet, it is obvious that the tubes would suffer a longitudinal *tensile* stress throughout the length BCD, the stress being slightly greater at the crown than at the springing.

EXAMPLE 3.—It may perhaps be worth while to work out the calculation for a segmental rib, which would be the more usual, and generally the better form for the arch. Thus, without altering the dimensions given in the last example, we may easily find (by similar triangles) that with a span of 120 feet, and a rise of 12 feet, the segmental rib would have a radius of 156 feet, that $\cot\frac{\theta}{2} = \frac{156-12}{60} = \frac{12}{5}$, that $\operatorname{cosec}\frac{\theta}{2} = \frac{13}{5}$, and that $\sin\frac{\theta}{2} = \frac{5}{13}$.

We may again assume a total load W of 120 tons, but if the line of pressure is to be a segmental curve passing through B, C, and D, the load upon the half span must have its centre of gravity in the same vertical line with the point G, which is the intersection of the tangents BG and CG.

With this distribution of load the horizontal thrust at the crown will be—

$$H = \tfrac{120}{2} \times \tfrac{12}{5} = 144 \text{ tons}$$

while at the springing the inclined thrust will be—

$$C_B = H \sec\frac{\theta}{2} = \frac{W}{2} \operatorname{cosec}\frac{\theta}{2} = 60 \times \tfrac{13}{5} = 156 \text{ tons}$$

When the pipe is charged under any given hydrostatic head, the axial fluid stress is easily calculated, and the compressive or tensile stress in the arched tube may readily be found by simple subtraction as before.

It may be noted, however, that under a head of about 396 feet (measured at B) the axial fluid stress would be 156 tons, and the tube would exert no thrust at the springing. This corresponds with the calculation of the displacing force given in Art. 10, where it was found that the total lifting force, or the vertical resultant of all the unbalanced fluid pressures distributed round the arc BCD, would be expressed by $2P \sin\frac{\theta}{2}$; for in this case P = 156 tons, and the lifting force would therefore be—

$$2 \times 156 \times \tfrac{5}{13} = 120 \text{ tons}$$

which is the weight of the bridge.

We have not, hitherto, taken into account any bending stresses which may be due either to change of temperature, or to the initial adjustment of the rib and its subsequent deformation under the imposed load. The calculation would of course be incomplete if it did not include these matters, but in regard to them it is only necessary to remark that the deformation of the rib due to fluid pressure (or displacing force) would have to be considered alongside the deformation due to load only, and would of course be a deformation in the opposite direction. Thus, if the stress Q should be found to be very small, the deformation of the rib due to the simultaneous imposition of load and hydraulic pressure would also be very small; and if Q were a tensile stress, the deformation would take place in a direction opposite to that which is usually provided for in calculating the stresses in arched ribs.

Art. 26. Equilibrated Fluid Arches.—The tubular arches considered in the last article were not wholly fluid arches; for although the fluid axial stress contributed largely towards the resistance of the arch, and might be sufficient to meet the greatest thrust at any point, yet the load could never be equilibrated by the fluid arch alone, because the fluid stress P was uniform throughout the arc (or nearly so), while the arch thrust C varies with the secant of the angle of inclination or sec ϕ.

But for any given distribution of load it is obviously possible (in theory at least) to design an equilibrated fluid arch which shall carry the whole load and leave *no* longitudinal stress in the tubes themselves. This would be accomplished, in fact, by making P a slightly varying quantity, and everywhere equal to H sec ϕ, which can very easily be done by making the arched tube with a varying diameter, increasing slightly from the crown to the springing.

Thus, for instance, in the case of a uniform load, the parabolic curve would evidently be the true curve for an equilibrated fluid arch; and if the fluid pressure p is constant, it will only remain to design the arched tube with a varying diameter d equal to $d_0\sqrt{\sec\phi}$, in which d_0 is the diameter of the tube at the crown of the arch.

Then the axial fluid stress at the crown will be $P_0 = pd_0{}^2\frac{\pi}{4}$, and at any other point it will be $P = P_0 \sec\phi$. If, then, the fluid pressure is so adjusted to the load that $P_0 = H = W\frac{L}{8D}$, it must follow that at all points the thrust C due to that load will be exactly met and resisted by the fluid stress P; and consequently the longitudinal stress in the metal of the tube will be everywhere, Q = nothing.

For example, the parabolic tube of Fig. 25 would become an equilibrated fluid arch if the tube were made with a diameter of 36 inches at the crown, increasing gradually to $36 \times \sqrt{1 \cdot 077}$, or say $37\frac{3}{8}$ inches at the springing; and when the tubes are charged with a uniform pressure equivalent to a head of 381 feet, the load of 120 tons will then be carried wholly by the fluid arch.

When the pressure is not absolutely uniform, but varies with the altitude of different points in the arch, the correction that is needed can be so easily applied that it would be superfluous to express it in mathematical terms.

A parabolic tubular arch designed in this way will evidently be an equilibrated fluid arch *for one particular load*—that is to say, the entire load will be carried by the internal arch of water, and not at all by the exterior envelope of steel, when the total load has the particular value $W_0 = P_0 \frac{8D}{L}$; and a little consideration will make it also evident that when the actual (uniform) load W has any value greater or smaller than W_0, the difference $W - W_0$ will constitute a uniformly distributed load that has to be carried by the steel arch in the ordinary way.

In this case, therefore, the direct compressive stress in the metal arch will be—

$$Q = C - P = (W - W_0)\frac{L}{8D} \cdot \sec \phi \quad . \quad . \quad . \quad . \quad (5)$$

If W is less than W_0, the stress Q will be negative or a tensile stress, equivalent at all points to the tensile stress in the chain of a parabolic suspension bridge under a uniform load.

In like manner we might proceed to deal with any other known distribution of load, and to trace, by the same principles, the corresponding form and proportions of the equilibrated fluid arch. Indeed, it would be quite easy to invent an almost endless series of structures which might be called fluid arches, or bowstrings, or polygonal trusses; for if any compression member in any such structure were formed as a tube filled with fluid under pressure, the function that would usually be performed by the compressive stress in that member would then be performed *in certain measure* by the axial fluid stress.

A particular case, which might be mentioned as possessing some interest in applied mechanics, is that in which the load consists only in the weight of the fluid itself, together with the weight of a tube strong enough to contain it. The curve of such a fluid arch, when equilibrated under its own weight, may readily be found by known mechanical principles, and might be called perhaps the fluid catenary. The construction of the curve becomes simple enough when the fluid is an elastic gas, such as compressed air, and is briefly described in the Appendix. The co-ordinates of the curve for an arch of compressed air contained in steel tubes devoid of compressive stress, are worked out in the Appendix up to a span of 1400 feet.

Such problems, however, do not properly belong to the subject-matter of this handbook. For practical men the question will possess an interest, not as a mechanical paradox, but as illustrating the action of forces that must sometimes be reckoned with in ordinary practice.

Keeping the last-named object in view, it may be worth while to revert for a moment to the case of a segmental arch or loaded segmental bend. If the external forces applied to the back of a segmental arch consist only of vertical forces or loads, the case will be analogous to that of the parabolic rib first referred to in this article, and *mutatis mutandis* may be treated in the same way, the varying diameter d being always made proportional to $\sqrt{\sec \phi}$; but, of course, a segmental tube *of uniform diameter* may become an equilibrated fluid arch if the external forces are applied in a *radial* direction opposite to the direction of the unbalanced internal pressure or displacing force, whose value per lineal foot of arc is $n = \frac{P}{R}$. When the external pressure has a uniform intensity equal to n, it will be exactly equilibrated by the fluid arch, and when it has any value n_e greater or less than n, the difference $n_e - n$ will evidently constitute a positive or negative normal pressure which must be borne by the tube itself. If the external pressure n_e is greater than n, the unsupported load $n_e - n$ may be carried by the tubular arch, if the tube is abutted upon unyielding skew-backs as at *B* and *D* in Fig. 25; and the compressive stress in the metal tube will evidently be—

$$Q = Rn_e - P \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

On the other hand, if n is greater than n_e, the unbalanced lifting force or displacing force $n - n_e$ may be borne by the tube in direct tension if it is *anchored* to fixed abutments at *B* and *D*, and the tensile stress will be given by the negative value of Q in the same expression.

Lastly, if the ends of the tube are neither abutted upon, nor anchored down to fixed abutments, but are carried in sliding expansion joints, as in Fig. 19, the tube cannot act as an arch, nor as a curved tie, and any excess of radial pressure in either direction must be borne by the tube as a girder. And whether the difference $n_e - n$ be positive or negative, the bending moments will have the values given in Art. 14.

Art. 27. Summary of Results.—These calculations of arch stresses, carried out upon familiar principles, have served to confirm and illustrate the results found in Chapter III. In applying those results, we have to keep in mind that there are four sets of forces or stresses concerned in the equilibrium of a curved pipe, viz.:—

The external load W, or normal pressure n_e;
The axial stress P in the fluid;
The axial stress Q in the pipe;
The resistance of the pipe to transverse bending;

and any one of these may be calculated when the other three are known.

Thus, in the curved tube of Fig. 19, if there is no elastic deflection, we have—

$$P + Q = Rn^e$$

and if Q is nothing, then $P = Rn$, from which we may calculate either the pressure p, or the radius R, or the external load *ne*, when the other quantities are given, or are in any way deducible from the known character of the surrounding conditions.

CHAPTER VI.

THE STABILITY OF A LINE OF PIPE.

Art. 28. Continuous Lines of Flanged or Riveted Pipe.—It was remarked at the outset that the displacing forces which have been examined in the last three chapters will only exhibit their action under certain conditions, and that in a great many cases their effects will not be manifested in any visible displacement of the pipe. We must, therefore, consider, in each individual case, how the displacing force is to be met, and how its action may be influenced by surrounding conditions.

If we consider the metal pipe apart from its fluid contents, we must admit that *every* bend, without exception, will be subjected to unbalanced internal pressures. But we have seen that the buckling tendency in a straight pipe, and the displacing force in a curved pipe, depend really upon the axial stress or force $(P + Q)$ transmitted through the column as a whole; and that they become absolutely annulled when $P + Q =$ zero, or when $- Q = P$.

Thus, if the straight steam-pipe AB in Fig. 22 were attached to the bend BCD by an ordinary flanged joint, it would be certainly capable of acting as a horizontal tie, which might relieve the cantilever BE of the whole bending stress due to the thrust H. It is obvious, indeed, that before any considerable displacement of the cantilever could take place, the pipe AB *would* then come into action as a tie, and would take up a portion or perhaps the whole of the axial thrust H. There would then be no buckling tendency in the pipe AB, and no further displacement of the bend would be possible.

It is easy enough, therefore, to see that in any *continuous* line of riveted or flange-jointed pipe, forming a series of bends and straight lengths, every part of the pipe might be maintained in equilibrium by the longitudinal tensile stress running continuously through straights and bends, and there would then be no lateral displacement anywhere.

In such cases, it would be reasonable to make the rivet strength, at the circumferential joints of a riveted pipe, sufficient to withstand at least the longitudinal pull $Q = -P$; and to make the tensile strength of the flange bolts sufficient to withstand the same pull in addition to the tension which may be requisite to ensure tightness at the joint.

And, in designing these joints, it would not be unreasonable to make some further provision for the additional stresses that might result from changes of temperature. In practice, however, the pipe-line will not always be constructed in this continuous form, and we may therefore proceed to trace the further application of the results that have been obtained in previous chapters.

Art. 29. Pipes Fitted with Expansion Joints.—To avoid the stresses that might result from changes of temperature, it will sometimes be necessary to insert these joints; and we must then remember that we have sacrificed the longitudinal strength of the pipe, and that the axial fluid stress, which may often amount to some hundreds of tons, will have to be transmitted through the column, producing in all the adjacent bends the displacing forces that have been calculated in Chapter III., and in the straight lengths the buckling tendency described in Chapter IV. The displacing force must now be resisted at each bend by one or the other of two possible methods. We may apply to the outer circumference of the bend a distributed normal pressure or reaction n_e, equal and opposite to the distributed displacing force n as calculated in Arts. 9, 10, and 11; or we may utilize the transverse strength of the pipe, employing it as a girder to transmit the displacing force to certain points of support. In the latter case, the bending stresses will be those calculated in Arts. 14 and 15; and at the chosen points of support it will, of course, be necessary to provide in some way for the requisite supporting reactions.

Art. 30. Lines of Socket-jointed Pipe.—The practical application of these theorems to lines of socket pipe is obviously surrounded by many uncertainties, because so little is known of the various resistances which may here be combined. Referring to the list of these resistances, as noted down in Art. 27, we may perhaps eliminate at once the transverse resistance of the pipe, as this can hardly be worth much when the joints are made in the usual way, with gaskin or some kind of elastic packing; and we are then left with two resistances which may be opposed to the displacing force, namely, the external normal pressure n_e and the tensile stress Q.

In most cases the pipes will be buried below ground, and however small may be the tensile strength of the lead joints, the earth pressure may perhaps suffice to contribute the requisite reaction n_e.

When, however, the pipe-line takes a bend in the vertical plane, with a curvature convex upwards, the earth pressure may be limited to the weight of filling above the pipe. Then if w denotes the total load per foot lineal, including the weight of the filling and the weight of the pipe itself with its fluid contents, we may find approximately the tensile stress, or—

$$\mathrm{Q} = \mathrm{P} - \mathrm{R}w \quad . \quad . \quad . \quad . \quad . \qquad . \quad (1)$$

Or, again, if it is desired to limit the tensile stress Q to some quantity which shall not exceed the safe working stress for the lead joints, we may denote that quantity by Q_j, and then the requisite load for a given radius of curvature R will be—

$$w = \frac{\mathrm{P} - \mathrm{Q}_j}{\mathrm{R}} \qquad . \quad . \quad (2)$$

or the safe radius of curvature for a given load will be—

$$\mathrm{R} = \frac{\mathrm{P} - \mathrm{Q}_j}{w} \quad . \qquad . \quad . \quad (3)$$

The tensile strength of lead socket joints is a matter which probably deserves more attention than has been commonly given to it, for in some cases the stress Q must be very considerable. The author is not aware of any experiments bearing directly on the question, but a few tests have been made at the laboratory of University College, Dundee, and the results are given in the Appendix.

The tests were made with a 50-ton Wicksteed machine, and include many of the usual forms of lead joint suitable for pipes of large diameter; but of necessity the sockets were small in diameter on account of the limited capacity of the machine. It seems probable, however, that in sockets of larger size the strength would be nearly proportional to the circumference measured at the line of shear.

So far as these tests have been carried by the author, they show a rather wide difference in the tensile strength of different forms of joint. When the pipe is made with a plain cylindrical end, without any shoulder, the resistance is comparatively small, and varies from 3 to 3½ cwt. per square inch of the area of contact.

But the resistance is considerably increased when the pipe is formed with a shoulder at the end; and even when the socket is bored to a smooth cylindrical surface, the lead refuses to be driven out bodily, and the joint only gives way by a slow shearing of the lead, leaving a thin lining along the whole length of the socket. This at least was found to be the case when the shoulder was slightly smaller in diameter than the bore of the socket, as it generally would be in practice.

Sockets formed with a recess did not always show an advantage over those which were cylindrical in the bore; but in this case again the tensile strength seemed to depend mainly on the presence or absence of a shoulder at the end of the pipe.

The employment of gaskin behind the lead appeared to be attended generally with some loss of tensile strength.

Art. 30a. Joints Turned and Bored.—In some instances the pipes of a buried water-main have been laid with turned and bored joints. If such a joint were made to fit very tightly, it is possible to conceive that it might possess some little transverse strength. But it is difficult to imagine that its *longitudinal* tensile strength could resist anything more than a very small fraction of the axial force P. The joint is not designed to resist any considerable pull of this kind; and at every bend the displacing force must be borne almost wholly by the reaction of the earth filling.

If we take Q_j = nothing, then, at the vertical bends we shall have, for the load that would just equilibrate the internal fluid pressure—

$$w = \frac{P}{R} \quad . \quad . \quad . \quad (4)$$

or, for the limiting radius of curvature—

$$R = \frac{P}{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In either case this would give us a factor of safety equal to unity only, and the values of w and R as above given would have to be multiplied by the factor that might be deemed reasonably sufficient to secure the stability of the pipe-line—due consideration being given to the contingency of an increase in the hydraulic pressure, such as might be due to "water-hammer" or other causes.

CHAPTER VII.

HYDROSTATIC PRESSURE OF VARYING INTENSITY.

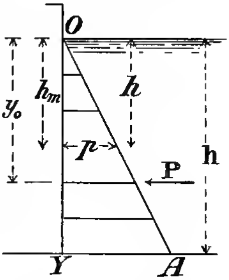
Art. 31. The Distribution of Hydrostatic Pressure.—The propositions given in Chapter II. will, of course, only hold good when the pressure is of uniform or nearly uniform intensity, and the calculations must be made quite differently when the fluid pressure is due to the weight of the fluid itself; for in that case we shall have to deal with a pressure of varying intensity $p = \gamma h$.

It will not be necessary here to demonstrate again the fact that p is exactly proportional to the depth h below the free surface, but we must keep that fact in view throughout.[1]

In every case the direction of the pressure will, of course, be at right angles to the surface of the solid boundary wall; and taking first the simple case of a *vertical* wall or dam exposed to hydrostatic pressure upon its plane surface, we can very easily illustrate the distribution of the *horizontal* pressures by means of a diagram.

Thus, in Fig. 26 let O be the water-mark upon the face of the vertical wall OY, the maximum depth of water being $OY = \mathrm{h}$; and let the horizontal line AY represent, to a certain scale, the maximum horizontal pressure $\gamma\mathrm{h}$. Then, drawing the straight line OA, the horizontal ordinates in the triangular diagram OAY will represent to the same scale the pressure p at any variable depth h below the free surface. The diagram will serve to remind us how the water pressure increases regularly in *intensity* as we go from the surface downwards.

FIG. 26



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[1] For the student entering freshly upon this part of the subject, it might be well to read again the elementary remarks of Art. 3.

We must remember, however, that in this familiar diagram the ordinates do not represent forces, but only pressures per square inch; and in order to find the forces, or the actual magnitude of the pressures in pounds, we must take each individual element of the wall surface and multiply its area by the pressure p that takes effect upon that particular element. Thus, to represent the *forces,* we require to construct, not merely a diagram, but a stereographic model; and the model will evidently be a wedge whose *faces* are indicated by the lines OY and OA in the diagram of Fig. 26.

Art. 32. The Measurement of Hydrostatic Pressure.—In Fig. 27 let OX represent the breadth b of a rectangular wall or of a dock gate, and let OY represent the depth of water h down to the cill. Then the pressures on all parts of this rectangular wall surface may be represented by the simple wedge-shaped model, sketched in perspective in Fig. 27*a*.

FIG. 27

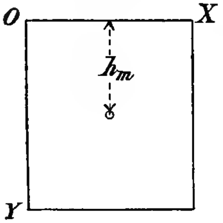
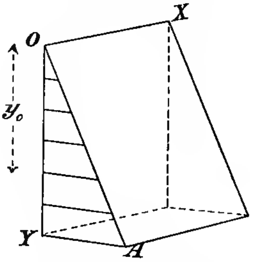


FIG. 27A



Suppose the model to be made of wood in which the fibre lies parallel to AY. The three dimensions of such a model will be, say, length in the direction AY, breadth OX, and depth OY. In any longitudinal fibre of the wedge, parallel to AY, the length will represent the pressure per square inch, while its sectional area will be the small element of wall surface on which that pressure acts. The cubic contents of the fibre will represent the actual pressure upon that element of surface, and the cubic contents of the whole wedge will therefore represent the total pressure P upon the whole surface of the wall.

The stereographic illustration will be applicable to all cases, but in this very simple example we might have done without it; for it is obvious that the *mean* pressure will be exactly one-half of the maximum pressure γh, so that the total pressure must be—

$$P = bh \times \frac{\gamma h}{2} = \frac{\gamma b h^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

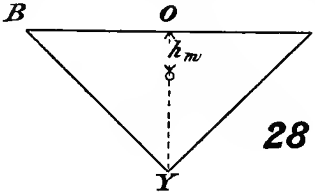
and the mean pressure will be that which takes effect at one-half of the maximum depth, or at a head—

$$h_m = \frac{h}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

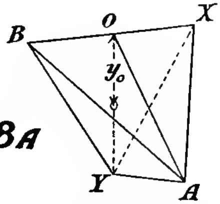
In like manner, we can find the total pressure upon a vertical plane of any other figure, multiplying the whole area by a certain mean pressure; and in the model the mean pressure will be equivalent to the mean length of fibre. The cubic contents of the wedge will always be a measure of the total pressure P, and the cubic contents are equal to area of wall surface multiplied by mean length of fibre. In consequence it may be mathematically shown that the mean pressure is the pressure which takes effect at the centre of gravity of the wall surface; so that the mean head h_m is in all cases the head of water measured down from the surface to this particular spot on the wall;[1] and if A denotes the area of the submerged surface, the total pressure will be, of course—

$$P = \gamma A h_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It may be useful to note down a few cases of common occurrence, remembering that in Art. 4, where the centre of gravity was determined, its position was given by the depth y below the *upper edge of the figure*, which may not always coincide with the water-level.



28



28A

In Fig. 28, the triangle *BXY* may be taken to represent the

---

[1] If x and y are the co-ordinates of any point on the surface of the wall, the general expression will be—

$$h_m = \frac{\int yx \, . \, dy}{\int x \, . \, dy} \quad . \qquad (3)$$

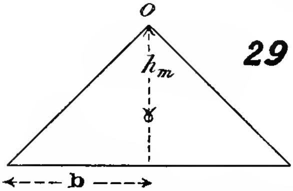
cross-section of a trough whose ends are closed by vertical walls.

Taking *BOX* as the water-line, the model will have the pyramidal form sketched in Fig. 28*a*. If b denotes the half-breadth *OX*, and h the depth *OY*, the submerged area will be $A = b\text{h}$; and the mean head h_m will be one-third of the depth h. Therefore, the pressure on the whole surface *BXY* will be—

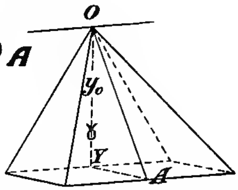
$$P = b\text{h} \times \frac{\gamma \text{h}}{3} = \frac{\gamma b \text{h}^2}{3} \quad . \quad . \quad . \quad . \quad (5)$$

If the triangular surface were inverted as in Fig. 29, where the apex of the triangle is supposed to coincide with the water-level, and where b denotes the half-breadth at the base, the area would be $A = b\text{h}$ as before; but the mean head would be $h_m = \frac{2}{3}\text{h}$, and therefore the total pressure is—

$$P = b\text{h} \times \tfrac{2}{3}\gamma \text{h} = \tfrac{2}{3}\gamma b \text{h}^2 \quad . \quad . \quad . \quad (6)$$



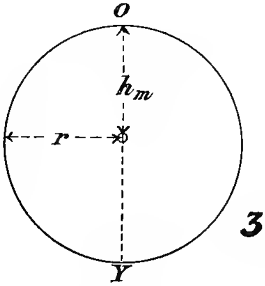
29



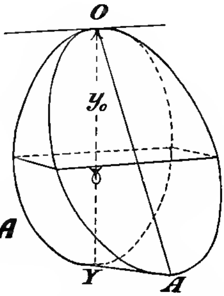
29 A

For a circular surface whose upper edge just touches the water-line, as in Fig. 30, we shall, of course, have $A = \pi r^2$, in which r is the radius of the circle; and as $h_m = r$, we have very simply—

$$P = \pi \gamma r^3 \quad . \quad . \quad . \quad . \quad . \quad (7)$$



30

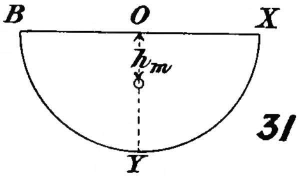


30A

But in the case of a semicircular trough or channel filled with water up to the diametral line *BOX* in Fig. 31, and closed at the end, the semicircular end wall would have the area $A = r^2\frac{\pi}{2}$, in which the radius is denoted by $r = b = h$.

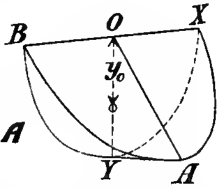
The mean head would be $h_m = \frac{4r}{3\pi}$; and the total pressure would therefore be—

$$P = \gamma A h_m = \tfrac{2}{3}\gamma r^3 \qquad . \quad . \quad . \quad . \quad . \quad (8)$$



31

31A



For a figure of flat segmental form, such as the cross-section of an invert, the exact area is given in Art. 4, as well as the vertical depth of its centre of gravity below the centre of the segmental arc; but in such cases we should introduce very little error if we assumed the curve to be parabolic, and the calculation would then be very simple.

If h is the depth, and b the half-breadth at the water surface, the area will be $A = \frac{4}{3}bh$; and as the mean head is $h_m = \frac{2}{5}h$, the total pressure when the invert is filled to the springing, or to any less height h, will be approximately—

$$P = \tfrac{8}{15}\gamma b h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This may be sufficiently correct for a very flat invert; but the multiplier $\frac{8}{15}$ will evidently increase with an increased depth of invert, and will become $\frac{10}{15}$ when the invert is semicircular. If h is greater than about $\frac{1}{5}$ or $\frac{1}{4}$ of b, it will be better to use the correct formula for a segmental area.

Art. 33. The Centre of Action.—The wedge-shaped solids, sketched alongside the several figures, will serve to illustrate the distribution of *horizontal* pressures upon the *vertical* surface in each case; and of course the centre of action would, in each case, be the same if the pressures were vertical and the surfaces horizontal. By a slight exercise of mental gymnastics, we can imagine all the wall surfaces to be laid flat, with the wedges upon them—the vertical length of each fibre (parallel to AY) would then be a measure of its weight, and the centre of action of the distributed pressures could be no other than the centre of gravity of the wedge itself.[1]

Thus for the regular figures illustrated in the last article, the vertical distance y_0 from the water surface to the centre of action of the horizontal pressure will be as follows—

For the rectangle (Fig. 27), $y_0 = \frac{2}{3}h$
„ triangle (Fig. 28), $y_0 = \frac{1}{2}h$
„ triangle (Fig. 29), $y_0 = \frac{3}{4}h$
„ circle (Fig. 30), $y_0 = \frac{5}{4}r = \frac{5}{8}d$
„ semicircle (Fig. 31), $y_0 = \frac{3}{16}\pi r = 0{\cdot}589r$

If the flat segmental area of an invert is assumed to have the parabolic form, we may take $y_0 = \frac{4}{7}h$ as the vertical depth from the chord line.

The moment of the horizontal pressure about the water-line *BOX* as an axis, will be $M = Py_0$ in each case. Thus, if the semicircle of Fig. 29 represents a door or flap hinged at the water-level *BOX*, the hydrostatic pressure would exert a turning moment—

$$M = Py_0 = \tfrac{2}{3}\gamma r^3 \times 0{\cdot}589r = 0{\cdot}3926\gamma r^4 \quad . \quad . \quad (11)$$

Art. 34. Vertical Walls of Irregular Figure.—When a dam crosses a valley or channel of irregular section, there will be no difficulty in calculating the total horizontal pressure upon the whole side of the dam; for if we divide the irregular area of the vertical plane into a number of regular areas A_1, A_2, A_3, etc., we can multiply each area by its proper mean pressure γh_m, and adding up all the products, we have—

$$P = P_1 + P_2 + P_3, \text{ etc.} = \gamma\Sigma(Ah_m) \quad . \quad . \quad . \quad (12)$$

The summation of products Ah_m will be equivalent to the quantity $\int yx \, . \, dy$ in formula (10); and we may take one example to illustrate the calculation.

Example.—Let Fig. 32 represent the elevation of a dock caisson placed across an entrance whose side walls are battered at 1 in 6, while the floor is formed by a slightly dished invert, for which we shall assume the parabolic form.

Let the width of invert *FC* be 60 feet, and suppose the depth of water *CF* to be 24 feet at the springing, increasing to $OY_1 = 27\frac{1}{2}$ feet at the centre, so that $YY_1 = 3\frac{1}{2}$ feet. On the height of 24 feet, the batter *BC* will be 4 feet, and the total width at the water-level *BE* will be 68 feet.

---

[1] The general expression will be—

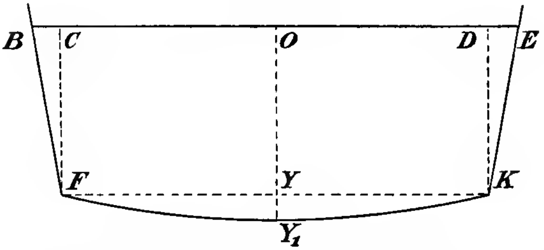
$$y_0 = \frac{\int y^2 x \, . \, dy}{\int yx \, . \, dy} \quad . \quad . \quad . \quad . \quad (10)$$

It is required first to calculate the total horizontal pressure P upon the side of the caisson.

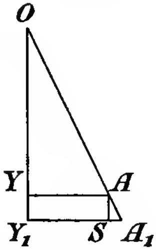
For this purpose we may divide the figure into three areas, of which A_1 shall be the rectangle *CDFK*, A_2 shall include the two triangles *BCF* and *DEK*, and A_3 shall be the segment FKY_1.

The area A_1 will be $60 \times 24 = 1440$ square feet, and its mean head will be $h_m = \frac{24}{2} = 12$ feet.

FIG. 32



32 A



The area A_2 will be $4 \times 24 = 96$ square feet, and its mean head $h_m = \frac{24}{3} = 8$ feet.

Lastly, $A_3 = 60 \times \frac{2}{3} \times 3{\cdot}5 = 140$ square feet, and for this segment the mean head (from the water-line) will be $h_m = 24 + (\frac{2}{5} \times 3{\cdot}5) = 25{\cdot}4$ feet. Consequently—

$$\begin{aligned} P_1 &= 1440 \times 12\gamma &&= 17{,}280\gamma \\ P_2 &= 96 \times 8\gamma &&= 768\gamma \\ P_3 &= 140 \times 25{\cdot}4\gamma &&= 3{,}556\gamma \\ \hline P &= P_1 + P_2 + P_3 &&= 21{,}604\gamma \end{aligned}$$

For the weight of the fluid we may follow salt water practice, making $\gamma = \frac{1}{35}$ ton, and therefore—

$$P = \frac{21{,}604}{35} = 617{\cdot}25 \text{ tons.}$$

If now we wish to find the centre of action of this distributed pressure, or to determine its vertical distance Y_0 below the point *O*, we must calculate the total moment M of these forces, *i.e.* their moment about *BOE* as an axis, and then we shall have $Y_0 = \frac{M}{P}$.

The moment M will be equivalent to the expression $\int y^2 x \,.\, dy$ in formula (10) multiplied by γ; and it may be regarded as the sum of the moments M_1, M_2, and M_3, due to the fluid pressures P_1, P_2, and P_3 upon the three areas. In calculating those moments we shall have to be a little cautious about the centres of action, but we can keep an eye on the wedge-shaped model, whose end elevation is sketched in Fig. 32*a*.

For the pressure upon the rectangular area A_1 the centre of action will be at the depth $y_0 = \frac{2}{3} \times 24 = 16$ feet below the water-level; and it is obvious that the moment of that pressure will be simply $M_1 = P_1 \times 16$ feet.

Also, it is evident that each of the triangles included in A_2 will subtend a certain portion of the wedge-shaped solid, having the form of a pyramid whose centre of gravity will lie at the depth $y_0 = \frac{24}{2} = 12$ feet below the water-level.

But when we come to the segmental area, and consider the shape of the corresponding part in the model, we see that the block AYY_1A_1 forms a sort of *truncated* wedge—a solid whose centre of gravity we have not yet found. That point will not coincide with the centre of gravity of the segment (which is 1·4 feet below the chord line), nor with the centre of gravity of the wedge ASA_1, whose vertical depth below the chord is $\frac{4}{7} \times 3{\cdot}5$ or 2 feet. But knowing the position of those centres, we can cut up the solid into two blocks, and find the moment of each of them. In the slice AYY_1S, the uniform length of fibre AY represents a uniformly distributed pressure, whose intensity is 24γ, whose magnitude is $24\gamma \times 140$ square feet $= 3360\gamma$, and whose centre of action is 1·4 feet below the chord.

In the little wedge ASA_1, the *mean* length of fibre will represent a mean intensity of pressure equal to $1{\cdot}4\gamma$, its cubic contents will represent an unevenly distributed force, having the additional magnitude of $1{\cdot}4\gamma \times 140$ square feet $= 196\gamma$; and the centre of action of this little force will be $\frac{4}{7} \times 3{\cdot}5 = 2$ feet below the chord-line.

The figures will then work out as follows:—

$$
\begin{aligned}
M_1 &= P_1 \times 16 \text{ feet} = 17{,}280\gamma \times 16 = && 276{,}480\gamma \\
M_2 &= P_2 \times 12 \text{ feet} = 768\gamma \times 12 = && 9{,}216\gamma \\
M_3 &= \left\{ \begin{array}{l} 3360\gamma \times (24 + 1{\cdot}4) = 85{,}344 \\ \;\;196\gamma \times (24 + 2) \;\;= \;\;5{,}096 \end{array} \right\} = && 90{,}440\gamma \\
& && \overline{376{,}136\gamma} = M
\end{aligned}
$$

Finally—

$$
Y_0 = \frac{M}{P} = \frac{376{,}136\gamma}{21{,}604\gamma} = 17{\cdot}41 \text{ feet}
$$

The calculation of the moment M is analogous to that by which the moment of inertia is determined for a beam of irregular section; and the process, as illustrated in this example, can be applied to other forms of wall surface. In using it we have always to distinguish between the different centres of gravity, remembering that the centre of gravity of the wall surface is the centre of action of a *uniform pressure;* and serves also to determine the mean head under hydrostatic pressure; while the centre of gravity of the wedge or of the solid model gives the centre of action of the pressure of varying intensity $p = \gamma h$.

Art. 35. Surfaces submerged in Deep Water.—Whatever may be the irregular figure of a plane surface, the total hydrostatic pressure can easily be calculated by the method illustrated in the last example; but in finding the exact position of the centre of pressure, we meet with a little complexity when we come to any element of surface so situated that its upper edge is below the water-level—as, for instance, in the case of the invert area in Fig. 32.

Such cases will frequently occur in practice, and we may generally deal with them without resorting to a computation of moments.

Thus, for any of the regular figures illustrated in Art. 32, let h_a denote the head of water above the upper edge of the figure, *i.e.* above the point O; and to avoid confusion, let us write y_m (instead of h_m) to designate the depth of the centre of gravity of *surface* below the same point O, so that we may designate the mean head by $H_m = h_a + y_m$.

Also, we may take y_0 to have the values already given in Art. 33, and to denote as before the vertical distance from the point O to the centres of gravity of the *wedges* sketched in Art. 32, so that those centres of gravity will be at the depth $h_a + y_0$ below the water surface.

Then writing algebraically the summation of moments, which was worked out in the last example, we can find a general expression for the depth Y_0 of the centre of pressure below the water-level, for—

$$Y_0 = \frac{\Sigma \cdot M}{P} = h_a + y_m \frac{h_a + y_0}{h_a + y_m} \quad . \quad . \quad . \quad . \quad (13)$$

The centre of pressure is therefore not a point fixed in one position upon the surface, but travels upwards and downwards within certain limits as the water-level is raised or lowered. Its vertical depth below the upper edge of the figure varies between y_0 and y_m, attaining the maximum value y_0 when

$h_a = 0$, and approaching to the minimum value y_m as h_a approaches infinity.

EXAMPLE.—Near the bottom of a lock-gate a rectangular sluice opening is formed, having a width of 24 inches and a height of 36 inches. Then, supposing the water to stand 16 feet above the top of the opening, it is required to find the pressure and the exact position of the centre of pressure.

Referring to Fig. 27, we have $y_m = 18$ inches, and $H_m = 16 + 1\frac{1}{2} = 17\frac{1}{2}$ feet; so that in salt water $p = \gamma H_m = \frac{17\frac{1}{2}}{35} = 0{\cdot}5$ ton per square foot, while $A = 2 \times 3 = 6$ square feet. Therefore, $P = 6 \times 0{\cdot}5 = 3$ tons.

By Art. 33 we have $y_0 = \frac{2}{3} \times 3$ feet $= 2$, and therefore, $Y_0 = 16 + 1{\cdot}5\left\{\frac{16 + 2}{16 + 1{\cdot}5}\right\} = 16 + 1{\cdot}543 = 17{\cdot}543$ feet.

Art. 36. Contrary Horizontal Pressures.—When the water stands on both sides of a dock-gate, but higher on one side than the other, the distribution of contrary pressures may readily be illustrated by two diagrams or two wedge-shaped models.

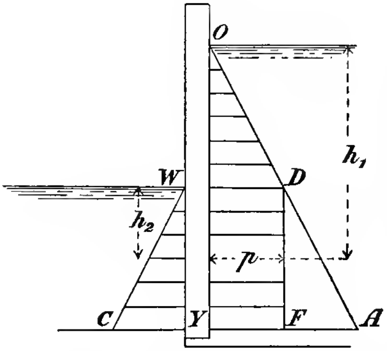
Thus, in Fig. 33, suppose that *O* marks the water-level on the inner side and *W* on the outer side of the gate. Then, at the level of the cill *Y*, we can set off on the opposite sides the ordinate *AY* to represent the pressure $\gamma \times \overline{OY}$, and the ordinate *CY* to represent the opposite pressure $\gamma \times \overline{WY}$; and drawing the straight lines *OA* and *WC*, the horizontal ordinates of these two triangles will give us the opposite pressures, which take effect on the two sides of the gate.

The effective pressure, or the pressure which has to be resisted by the gate as a whole and by its supports, will be the excess of the one pressure over the other, although the inner skin of the gate will, of course, be subjected to the entire pressure on that side. If h_1 denotes the depth of any point below the water-level *O*, and h_2 the depth of the same point below *W*, the effective pressure will evidently be—

$$p = p_1 - p_2 = \gamma(h_1 - h_2) \quad . \quad (14)$$

The subtraction of p_2 from p_1 may be done graphically upon the diagram, by setting off *AF* equal to *CY* and drawing

FIG. 33



the straight line *DF*, which must obviously be a vertical line parallel to *WY*. The diagram of effective pressures will then consist of the triangle *OWD* and the rectangle *WYFD;* for below the point *W* the effective pressure will be everywhere equal to *WD*, however deep the water may be.

The total effective pressure P and its centre of action are very easily found.

Example.—Let the breadth of a rectangular gate be $b = 35$ feet, height $OY = 24$ feet, and height $WY = 12$ feet. Then the upper part of the gate, submerged on one side only, will have an area of $35 \times 12 = 420$ square feet, with a mean head of 6 feet, and the total pressure P_a will be $420 \times 6\gamma = 2520\gamma$. On the lower part of the gate, entirely submerged, the effective pressure will evidently be twice as great, or $p = 12\gamma$; and as the area is again 420 square feet, the total pressure on this rectangular surface will be 5040γ, and may be denoted by P_b. The horizontal force exerted upon the whole gate and its supports will then be—

$$P = P_a + P_b = 7560\gamma, \text{ or say 216 tons}$$

(taking $\gamma = \frac{1}{35}$ ton as before).

The moments of these horizontal forces about the point *O* are quickly reckoned, and will stand as follows:—

$$\begin{aligned} M_a &= P_a \times \tfrac{2}{3} \times 12 \text{ feet} = 2520\gamma \times 8 = 20{,}160\gamma \\ M_b &= P_b \times (12 + \tfrac{12}{2}) = 5040\gamma \times 18 = \underline{90{,}720\gamma} \\ &\qquad\qquad\qquad\qquad\qquad\qquad 110{,}880\gamma = M \end{aligned}$$

The centre of action will therefore lie below the water-line by the vertical distance—

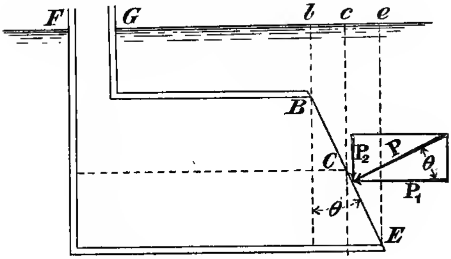
$$Y_0 = \frac{110{,}880}{7{,}560} = 14\tfrac{2}{3} \text{ feet}$$

Art. 37. Hydrostatic Pressure upon Inclined and upon Curved Surfaces.—Suppose a horizontal cylindrical pipe to be closed at one end by a flat cover, which is not hung vertically, but inclined to the vertical by any given angle θ as in Fig. 34; and let the interior of the pipe be kept free from water while the external surface of the inclined flap is submerged to a depth H_m measured to the centre of the pipe. As the centre of gravity of the elliptical surface will be in the centre of its figure, the mean head will always be the depth H_m whatever may be the inclination of the cover, or the length of the major axis *BE*. The actual pressure upon the cover will, of course, be at right angles to its inclined surface; but we may often have to consider the horizontal and the vertical components.

At any point whose vertical depth below the water surface is h, the pressure is $p = \gamma h$; and if a denotes a small element of the inclined surface, the pressure pa upon that element will have for its horizontal component the value $pa \cos \theta$. But the quantity $a \cos \theta$ represents the area (a_1) of the vertical projection of the same element of surface, and therefore the horizontal component of this element of the total pressure is simply pa_1 or $\gamma h a_1$.

As the same thing holds good for *every* element of the area, it is obvious that the entire horizontal thrust will be the same as though the end of the pipe had been closed by a vertical door. If A is the area of the inclined elliptical flap *BE*, and A_1 the area of its vertical projection (which is circular), we have $A = A_1 \sec \theta$, or $A_1 = A \cos \theta$; so that the horizontal thrust will be $P_1 = \gamma H_m A_1$, while the actual pressure at right angles to the flap will be $P = \gamma H_m A = \gamma H_m A_1 \sec \theta$.

FIG. 34



34 A

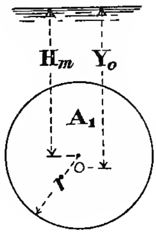
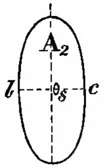


FIG. 34 B



Turning again to the pressure pa acting normally upon the small element of surface a, its vertical component will be $pa \sin \theta$. But the quantity $a \sin \theta$ represents the area (a_2) of the horizontal projection of the same element, so that the vertical component will be pa_2, or $\gamma h a_2$, which means simply that the vertical pressure, or load, upon the element a is equal to the weight of the column of water, or vertical fibre of water, standing directly over it; and this must obviously be true for every element of area, and for the whole area of the flap, whatever may be its inclination.

The horizontal projection of the whole surface will be the ellipse sketched below the section, and its minor axis $\overline{be}$ will be $2r \tan \theta$, so that the area of the projection will be $A_2 = A_1 \tan \theta$. The vertical column of water *BbeE* having this elliptical section, will have H_m for the mean height of all the vertical fibres; and the total vertical thrust will be $P_2 = \gamma H_m A_2 = \gamma H_m A_1 \tan \theta$.

The vertical force P_2, which is simply the weight of the skew-ended column *beBE*, will have its centre of action at the centre of gravity of that column, lying a little to the right of the centre-line *cC*. In the same way, the centre of pressure for the horizontal force P_1 will lie a little below the centre *C*; and its depth Y_0 below the water-level can readily be found by formula (13) of Art. 35. The exact position of the centre of pressure is found by a precisely similar calculation in both cases, and the lines of action intersect upon the inclined surface at the point which is itself the centre of action of the inclined force P.

EXAMPLE.—A horizontal pipe 4 feet in diameter is closed at the end by an elliptical flap hanging at an angle $\theta = 26°34'$ with the vertical, so that $\tan \theta = 0·50$, and $\sec \theta = 1·118$. Then, if the water stands 5 feet, for example, over the top of the pipe, we shall have a mean head $H_m = h_a + y_m = 5 + 2 = 7$ feet, and a mean pressure $p = \gamma H_m = \frac{7}{35} = 0·2$ ton per square foot.

The cross-section of the pipe, or the vertical projection of the flap, will have an area $A_1 = 4^2 \times ·7854 = 12·566$ square feet; and the horizontal thrust will be—

$$P_1 = 12·566 \times 0·2 = 2·513 \text{ tons}$$

Referring to Art. 33, we find for the circular area $y_0 = \frac{5}{4}r = \frac{5}{4} \times 2$ feet $= 2·5$ feet; and therefore the depth Y_0 from water-level to the centre of pressure will be given by formula (13)—

$$Y_0 = 5 + 2\left\{\frac{5 + 2·5}{5 + 2}\right\} = 5 + 2·14 = 7·14 \text{ feet}$$

In the horizontal projection, the minor axis being $2r \tan \theta$, gives $be = 4 \times 0·5 = 2$ feet; and the elliptical area $A_2 = 12·566 \times 0·5 = 6·283$ square feet. The vertical force, or the weight of the skew-ended column, is therefore $P_2 = 6·283 \times 0·2 = 1·256$ ton; and repeating the calculation for its centre of gravity, we find the distance $bs = 1·07$ feet. The lines of action of the forces P_1 and P_2 lie, therefore, at distances 0·14 feet below the centre, and 0·07 feet to the right of *cC*, intersecting each other upon the inclined surface at a point whose inclined distance from *C* will evidently be $0·14 \times \sec \theta$.

Lastly, the normal pressure whose resultant acts at this point will be—

$$P = P_1 \sec \theta = 2·513 \times 1·118 = 2·81 \text{ tons}$$

Let us now suppose the conditions to be exactly reversed, *i.e.* let the pipe be surrounded by atmospheric air, and filled with water until the water stands at the level *FG* in a stand-pipe.

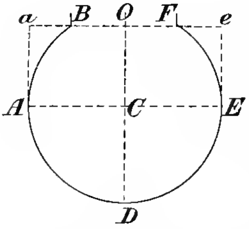
Then we shall have the same fluid pressure as before at each point in the flap, but acting in the opposite direction upon its internal surface, which is now an *overhanging* inclined plane.

Resolving then into horizontal and vertical components, it is evident that the upward pressure upon any element a will be equal to the weight of a water-fibre whose height is h and sectional area a_2. The whole buoyant force will therefore be equal to the weight of the liquid column *beBE*, and this distributed lifting force will have a varying intensity represented everywhere by the varying height h. Lastly, the centre of action of this vertical force, or centre of buoyancy, must obviously coincide with the centre of gravity of the skew-ended column of water which before stood over the flap, but has now been removed.

It is hardly necessary to say that the same principles can be extended to any series of surfaces inclined at different angles, and therefore to any surfaces of single or double curvature.

If, for example, the spherical bowl of Fig. 35 were filled with water up to the level *BOF*, the downward pressure upon the lower hemispherical surface *ADE* would be equal to the weight of the volume *aADEe* (partly spherical and partly cylindrical). But the lifting force upon the underside of the domed surface *AB* or *EF* would be equal to the liquid weight of the annular body whose section is *AaB* or *EeF*; and the algebraical sum of these opposite forces would, of course, be the weight of water in the bowl. Again, if we suppose the bowl to be empty and to be surrounded by water up to the level *a* . . . *e*, the upward pressure upon the lower hemisphere would be equal to the weight of liquid *aADEe*, while the liquid volume *AaB* would stand vertically over the domed surface all round; and the net result would be a buoyant force equal to the weight of water displaced, or the so-called "displacement" of the bowl.

FIG. 35



CHAPTER VIII.

FLOATING VESSELS.

Art. 38. Equilibrium of Floating Bodies in General.—It will not here be necessary to refer to the calculations which are commonly employed in shipbuilding practice; but the engineer has often to deal with floating vessels of quite special forms, and may have to design them for the accomplishment of objects which are not at all familiar to the shipbuilder.

Whether the vessel be a pontoon with a great top-heavy load, or whether it be a bridge caisson, a floating dock, or a diving machine, the principles which govern its equilibrium and its stability as a floating vessel will, of course, be the same in all cases, although their application will vary widely with the widely varying conditions. The questions we are concerned with will come under two headings, for the equilibrium of every floating body demands or implies the fulfilment of two sets of conditions—(1) the vessel must be supported by a buoyant force (or reaction) exactly equal to its own weight; (2) the forces must be so opposed to each other as to produce no upsetting moment; and in each case we have to consider not merely the condition of equilibrium, but also the disturbance and restoration of that condition, or, in other words, the question of the vessel's stability or instability, both as regards a rolling motion and a motion or oscillation in the vertical direction.

Art. 39. Buoyancy.—The buoyant or supporting force is evidently due in all cases to hydrostatic pressure upon the vessel's outer surface, for it cannot possibly be due to anything else. A little consideration will show that, whatever may be the form of a vessel floating in calm water, the horizontal components of that hydrostatic pressure will exactly balance each other, while the sum of all the vertical components will constitute a buoyant force whose magnitude is, in every case, to be measured by the cubic volume of the immersed portion of the vessel, or cubic displacement of the vessel, multiplied by γ.

In the case of a wall-sided and flat-bottomed vessel like that sketched in Fig. 36, it is obvious that the upward pressure $h\gamma$, taken over the whole area of the flat bottom, must have this value; and, as shown in Art. 37, the statement is equally true for dish-shaped vessels, and for vessels with coving sides; while a slight extension of the same illustration would show that it is also true when the body is totally submerged.

Therefore, if V represents the volume of the displaced water in cubic feet, we have in every case the buoyancy $B = V\gamma$.

The so-called "displacement" may sometimes be given in cubic feet (V), or more commonly in tons, when, of course, it means the same quantity as the buoyancy, or $\frac{V\gamma}{2240}$.

Art. 40. Equilibrium of the Vertical Forces.—If the vessel is to float at all, in any sense of the word, we must have $W = V\gamma$, where W denotes the weight of the vessel, including everything that she carries; and if the vessel *does* float, we may be sure that the equality $W = V\gamma$ subsists.

When the vessel floats with a considerable "free-board," possessing, therefore, a certain reserve of buoyancy, the equality is adjusted automatically by the vessel sinking to such a "draught" as will suffice to submerge the requisite volume. If W is increased by taking in cargo, or decreased by the gradual consumption of coal, V will be automatically increased or decreased in the same proportion. And if γ is increased or decreased by sailing from fresh into salt water, or back again, V will be decreased or increased in the inverse proportion. The equilibrium is therefore "stable" in ordinary cases, but the stability will depend upon the value of the increment ΔV corresponding to unit increment of draught, and in some special cases it may become very slight, or perhaps disappear altogether.

Art. 41. Vertical Motion under the Vertical Forces.—In a wall-sided vessel the increment of volume ΔV for 1 foot increment of draught, is obviously measured by the area of the horizontal section taken at the momentary water-line; and in vessels with "flaring" sides, or with "tumble-home" sides, the same area will measure the differential of volume with respect to a very small increment of draught. We may briefly consider three cases in which the stability is influenced by the value of the increment ΔV.

1. When a floating dock, of the old "Bermuda" type, swims with a light draught, so that the floor of the dock is above water, the increment ΔV is measured by the large area of the pontoon, and the equilibrium is stable in a very high degree; but it becomes more sensitive when the dock is lowered into the water, leaving only a reserve of buoyancy in the cellular sides, for now the increment ΔV will be measured by the area of the side walls only, and if their horizontal section were very small the equilibrium would be very sensitive.

2. When the increment ΔV vanishes, the stability vanishes also. Thus a closed vessel, with rigid sides and roof, may perhaps be intended to descend through the water, as in the case of a diving machine that was tried some years ago; but as soon as any vessel of this kind becomes totally submerged, its further descent is not attended with any increase of V, and as the density γ is practically constant, the buoyant force is an unchanging quantity. If the weight W is made great enough to sink her one inch below the surface, it must be a little greater than B, and however small may be the excess, the vessel must go to the bottom of the sea before the equilibrium can be restored; or if she starts from the bottom with an excess of buoyancy ever so small, it will suffice to carry her slowly to the surface. At any intermediate level the vessel is unmanageable, even with the aid of pneumatic appliances for the arbitrary adjustment of W and B.

3. When the increment ΔV becomes a negative quantity, the equilibrium becomes altogether unstable. This condition of things will be realized, for example, in any self-supporting diving machine which contains a pneumatic working chamber, open at the bottom like a diving-bell, as in the case of a submarine boat described to the author by one of her crew. For the displacement of such a pneumatic chamber is obviously the volume of the contained air, which varies inversely with the pressure, and therefore inversely with the depth. As the vessel descends under any slight excess of W over B, the excess becomes continually greater, and *vice versâ*, so that the vessel either mounts to the surface or sinks to the bottom of the sea with an ever-accelerating velocity. It was, in fact, found impossible to keep her at any intermediate level except by putting her into

the condition of a captive balloon, by means of sinkers and cables.

Art. 42. Lines of Action of the Vertical Forces.—Remembering that the downward force, or weight W, has for its resultant a vertical line passing through the centre of gravity of the floating body, it needs only to be remarked that the position of that centre of gravity is not always to be found without some difficulty. It may, of course, be determined by summating the moments of every element of the weight, including all the structural parts of the vessel as well as every part of her cargo; and although this would be a troublesome matter for the shipbuilder, it is not quite so troublesome in vessels of regular geometrical form. The shipbuilder's method of finding it experimentally by heeling the vessel has often been described, and is very well known, but in designing the vessels used by engineers an *à priori* calculation is often indispensable.

On the other hand, we can easily find the exact distribution of the buoyant pressure at all parts of the vessel's immersed body, and the exact line of action of the buoyant force, by direct measurement.

Whatever may be the shape of the immersed body, we have already seen (in Art. 37) that the pressure ap or $ah\gamma$ upon any element of surface a will have for its vertical component the quantity $a_2h\gamma$, in which a_2 is the horizontal projection of the inclined surface a; and it follows that the section of the floating vessel, with the water-line drawn across it as a datum, will itself constitute a diagram showing the varying intensity of the buoyant pressure per unit of area of the ship's horizontal section, while a model of the immersed body, or the vessel herself, will constitute the stereographic force-model.

Thus, taking Fig. 36*a* to represent the section of a vessel floating at rest in an inclined position, the immersed body will be a solid figure bounded by the water-plane as its upper boundary, and we may conceive the buoyant force as an unevenly distributed pressure acting vertically upwards with the varying intensities represented by the vertical ordinates intercepted by the bounding surfaces of the solid. For the height of each ordinate, or vertical fibre in the model, measured from the water-plane, represents the hydrostatic pressure, and the cubic contents of the fibre represent the vertical lifting force applied at its base.[1] This view of the distributed forces makes it obvious that the centre of action, or the so-called "centre of buoyancy," coincides with the centre of gravity of the displaced water, and makes it impossible to go wrong in applying the well-known principle to special cases.

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[1] It is sufficiently evident, by simple subtraction, that the same thing is true

Art. 43. Conditions of Equilibrium in Still Water.—Assuming the water to be calm and the vessel to be free from the action of any lateral forces (such as wind pressure), it will be obvious that there can be no equilibrium unless the lines of action of the opposed forces W and B coincide in one and the same vertical line. Thus, if Fig. 36 represents the cross-section of a rectangular pontoon in an upright position, the centre of buoyancy (for that position) will, of course, be situated at a point B in the vertical centre-line ACB, and if the pontoon is to float upright, it must be so loaded as to bring the centre of gravity to the same centre-line. For if the centre of gravity lay to the right or to the left of the centre, the forces W and B would exert a turning moment which would immediately cause the vessel to heel over to the right or to the left.

FIG. 36

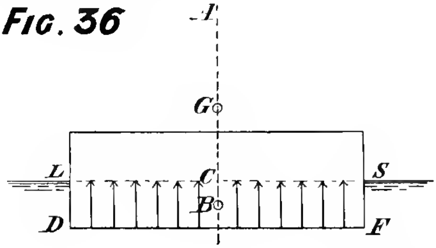
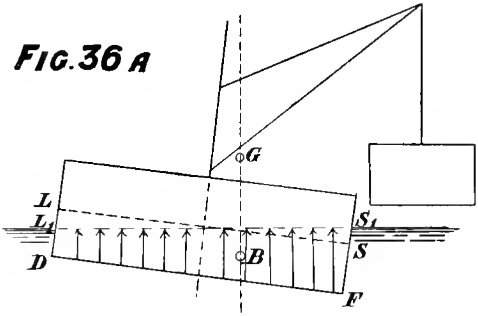


FIG. 36 A



If the pontoon, for any reason, heels over to an inclined position,

when any fibre is intercepted both above and below by the skin of the vessel, as, for example, in the left-hand corner of the figure.

as sketched in Fig. 36*a*, the wedge-shaped solid of displacement L_1DFS_1 will now have for its centre of buoyancy a point B_1, which must evidently lie to the right of the centre-line; and if the pontoon settles at rest in this inclined position, we may be sure that her centre of gravity lies somewhere upon the vertical line GB_1 drawn through the new centre of buoyancy. And it cannot lie to the right or to the left of that line for the same reason as before.

In either case, therefore, equilibrium can only be established when the centre of gravity G lies in the same vertical line with the centre of buoyancy B. When that condition is fulfilled, it still remains to be seen whether the equilibrium is stable or unstable; and this important question will be determined by observing what changes take place in the relative positions of G and B as the vessel rolls from the position of equilibrium to the one side or the other, and by noting whether such a change of position is attended by a "righting moment" or by an "upsetting moment."

Perhaps the most important questions for the shipbuilder are the stability of the vessel in the upright position of equilibrium, and the value of the righting moment for different angles of heel; but in the case of pontoons or lighters heavily loaded by excentric weights, it may be necessary to consider their stability in an inclined position of equilibrium.

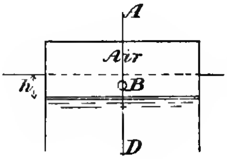
Art. 44. Stability of a Closed Vessel wholly Submerged.—In this case the centre of buoyancy is a fixed point in the interior of the vessel (coinciding with the centre of gravity of the submerged solid, whatever shape that solid may have), and does not move at all from its position as the vessel rolls in either direction; so that B becomes a virtual point of suspension. If the submerged vessel is symmetrically shaped about a vertical centre-line, and symmetrically loaded, she will either float upright or turn exactly upside down, according as the point G is situated higher or lower upon the centre-line than the point B. In the case of a submerged homogeneous solid G will, of course, coincide with B, and the body will possess no stability in any position.

Art. 45. Stability of a Floating Gas-holder or an Inflated Bridge Caisson.—Here we need only consider the question of stability in the upright position, and will accordingly assume that in Fig. 37 the weights are symmetrically distributed about the vertical centre-line *ABD,* and that the whole load is borne by the pneumatic pressure of the contained air or gas. That pressure must be equivalent to the hydraulic pressure $h\gamma$, due to the head h, which is the vertical height of the exterior water-line above the interior water-line; so that disregarding the trifling volume of water displaced by the thin plate sides, we may take the layer of uniform thickness h as constituting the whole displacement.

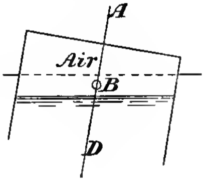
The centre of buoyancy B will, therefore, lie in the middle of that layer, halfway between the two water-levels, and in the centre of the vessel; and as the vessel rolls from side to side, the centre of buoyancy will remain in that position, so that here again the fixed centre of buoyancy is a virtual centre of suspension, and the vessel can only float upright when G is below B.

The question whether a bottomless caisson can be made to float safely *at all* upon the contained air will depend upon the distribution of the metal in the sides and in the cover, and also upon the total weight of the vessel, as the latter must govern the requisite thickness of the layer of displacement. If the cylinder has a great height as compared with its diameter, and is built of very thin plate, the requisite layer of displacement might perhaps be contained within a narrow zone quite near to the top of the cylinder, while the centre of gravity might lie far below, and the vessel would then float safely enough. To gain this end, it might sometimes be practicable to adjust the quantity of contained air so as to bring the caisson down to a suitable level, while still maintaining a sufficient height of freeboard; for the caisson might be unstable at one level, and yet stable enough when floating at a lower level. In the same way it is conceivable that a gas-holder let down to a very low level might float safely without the protection of any roller-guides; but during the

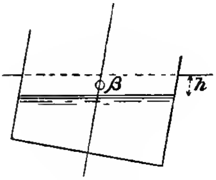
FIG. 37



37 A



37 B



process of "blowing up" the stability would decrease, and as soon as the centre of gravity is raised above the middle of the thin belt of displaced water, the gas-holder will topple over if the guides are not there to keep it up.

Art. 46. Stability of a Water-logged Vessel with Thin Sides.—If the vessel sketched in Fig. 37 were turned right over so as to float with its open end uppermost, and were partly filled with water, we should have a case very similar to the last.

The case may be considered in two ways. If we first take the entire submerged volume as representing the displacement and as measuring the buoyancy of the vessel, we must, of course, reckon the contained water as a part of the load; and we must then say that as the vessel rolls over to the inclined position of Fig. 37*b*, the centre of gravity and the centre of buoyancy will *both* travel simultaneously towards the right. But if we again neglect the displacement of the thin plates, we may obtain a simpler view of the forces that are really opposed. For if we take the layer of uniform thickness (h) as constituting the effective displacement by whose buoyancy the weight of the metal shell is supported, we shall arrive at the same result as by summating all the opposed vertical forces at all parts of the section. In other words, the net buoyant pressure, after deducting the water-load, is at all points the pressure $h\gamma$, and the centre of action of this uniformly distributed force is of course the point β in the middle of the layer.

If, therefore, we understand G to be the centre of gravity of *the shell*, it must follow that the vessel can only float upright when G is vertically below β, the latter point being again a virtual point of suspension.

As regards the position of the point G, it will be seen that its depth below the outer water-line will increase as the load increases, while the depth of the point β below that line is independent of the water-load. Thus a bucket floating at light draught, with only a little water inside it, will probably be unstable; but if water is allowed to flow gradually into it, the bucket will gradually descend, the point G descending at the same time, and the bucket will settle into the upright position before it finally sinks out of sight. In this case, as in the gas-holder, stability in the upright position depends upon the vertical depth of G below β.

The conditions would of course be altered if the bucket, or the gas-holder, were fitted with vertical bulkheads dividing the interior space into separate cells; but the effect of water-ballast taken into separate compartments must be considered later.

Art. 47. Stability of a Cylindrical Pontoon floating on its Side.—Let Fig. 38 be taken to represent the circular cross-section of a long pontoon, such as that which was used for the transport of Cleopatra's Needle, and let the diametral line *ACD* represent the centre line of the vessel when floating in the intended upright position. To fix the ideas, we may call this line the centre-line of her mast, and we will further suppose that the line is marked in red paint upon a transverse bulkhead of the vessel. The three centres, *C*, *G*, and *B*, will all be situated upon this vertical line, and we may suppose the point *B* to be marked upon the bulkhead at the exact centre of gravity of the segmental area *LDS*. The stability of the vessel in this position will depend upon the changes that take place in the relative positions of these centres as the pontoon rolls to an inclined position such as that sketched in Fig. 38*a*. The red paint line having swung into the inclined position *ACD*, the point B_1 will have moved away from that line, and will have described upon the bulkhead a part of the "curve of buoyancy," which is simply the path traced by the centre of buoyancy in its successive positions. The new segmental area L_1DS_1 will have exactly the same shape as the original area *LDS*; the distance CB_1 will be the same as before; and a vertical drawn through B_1 at right angles to the chord L_1S_1 will again pass through the centre of the circle *C*.

FIG. 38

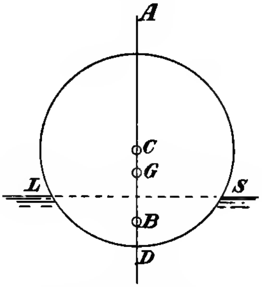
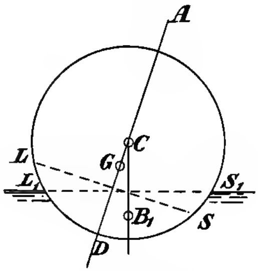


FIG. 38A



Hence, in this cylindrical form of vessel, the point *C* becomes a virtual point of suspension, and is a true "metacentre;" and the stability of the pontoon depends wholly on the question whether, in Fig. 38, the point G is below or above C. If it is below the metacentre C, any rolling of the vessel to the right or to the left will give rise to a righting moment whose value will be $W \times \overline{CG} \sin \theta$, and the pontoon will return to the upright position; but if G is above C, the vessel must turn over keel uppermost.

In each of the cases described in Arts. 44, 45, and 46, stability was only to be attained by placing G below B. Here that is not necessary, but stability depends upon the placing of G below *a new centre*, or "metacentre" C, which coincides, in this particular form, with the centre of the cylinder.

As compared with other and more usual forms, the cylindrical pontoon and the cigar ship possess the following distinctive properties:—

The curve of buoyancy is obviously the arc of a circle which might be continued all round the bulkhead.

Its radius of curvature CB will depend upon the depth to which the vessel is sunk by her loading, and becomes zero when she is totally submerged; but at all depths of draught the metacentre C retains the same position in the vessel, and also at all angles of heel.

Hence the cylindrical pontoon has only one possible position of stable equilibrium, provided that the cargo does not shift; while vessels of other forms may have two or more. Thus a flat-bottomed and flat-decked pontoon may find a position of stable equilibrium when floating keel upwards.

It also follows that the diagram of righting moments for varying angles of heel is in this case simply the curve of sines.

Art. 48. Stability of a Rectangular Pontoon in the upright position.—Let Fig. 39 represent the section of a pontoon which is rectangular in plan as well as in cross-section, and which is intended to float upright, her centre of gravity being situated somewhere upon the vertical centre-line ACD. Also let S_0L_0 represent the water-line, the centre of buoyancy being at the point B_0 in the middle of the immersed rectangle $L_0L_3FS_0$.

To observe what changes take place as the vessel rolls, we need not sketch the pontoon in a series of inclined positions, but may draw upon the figure as it stands, the lines S_1L_1, S_2L_2, etc., representing successive positions of the water-line as the vessel rolls over more and more to the right. With the heeling of the pontoon, the figure of the immersed section continually changes its shape, and the point B moves away to the right of the old centre-line, coming finally to the point B_3, which is the centre of gravity of the triangle L_3FS_3.

The shipbuilder's ordinary rules for finding the new positions of the centre of buoyancy would be as follows:—

1. When the vessel has rolled through any angle L_0CL_1, the wedge of emersion L_0CL_1 (rolled out of the water) must have the same volume as the wedge of immersion S_0CS_1 (rolled under water), because the total volume of displacement remains unaltered; and the water-lines L_1S_1, L_2S_2, etc., must be drawn accordingly, so as to make the pair of wedges equal in volume, whatever may be the angle of roll.

But in the rectangular pontoon it is obvious that this rule will be satisfied by drawing all the water-lines through the point C at the middle of the load water-line—so long at least as the lines do not travel beyond the limits of the vertical sides—*i.e.* so long as the vessel does not dip her deck under water nor roll her floor out of water.

2. Having found the centres of gravity, e and i, of the two wedges L_0CL_1 and S_0CS_1, the rule would be to draw the straight line ei; and then drawing a line through B_0 parallel to ei, we may set off a distance B_0B_1 proportional to the length ei, and equal to that length multiplied by the ratio $\frac{\text{volume of wedge}}{\text{total volume of displacement}}$.

But in the case of the rectangular pontoon, we can proceed very simply as follows, and with greater clearness as to results:—

Let b denote the half-breadth of the pontoon, and D the draught, so that the immersed section has an area $\mathrm{A} = 2b\mathrm{D}$; and let δ represent any vertical depth, such as L_0L_1, through which the water-mark has descended from the original water-mark L_0 on the side of the vessel. Then the volume of either wedge will be $\frac{b\delta}{2}$ multiplied by the length of the pontoon, and the ratio of that volume to the total displacement will be simply $\frac{\delta}{4\mathrm{D}}$.

The centre of gravity e of the wedge of emersion will lie to the left of the centre-line, at a distance of exactly $\frac{2}{3}b$, measured parallel to the floor, and below the water-line S_0L_0 at a distance of exactly $\frac{\delta}{3}$, measured parallel to ACD; while the centre i will lie at the same distances in the opposite directions.

It follows, therefore, that the centre of buoyancy will have moved to the right of B_0, in a direction parallel to the floor, through a distance $x = \frac{4}{3}b \times \frac{\delta}{4D} = \frac{b\delta}{3D}$, and at the same time will have moved upwards in a direction parallel to DCA, through a distance $y = \frac{2}{3}\delta \times \frac{\delta}{4D} = \frac{\delta^2}{6D}$.

These quantities x and y are, therefore, the co-ordinates of the curve of buoyancy, and taking for δ the successive values L_0L_1, L_0L_2, L_0L_3, we can fix at once the corresponding points, B_1, B_2, B_3, by the calculated series of co-ordinates, in which x is simply $\propto \delta$, and y is $\propto \delta^2$. The curve is, therefore, a parabola, and is very easily drawn.

Having fixed B_1, B_2, etc., we may draw through each of these points the line of action of the buoyant force. These lines, B_1m_1, B_2m_2, etc., will be drawn at right angles to the corresponding water-lines L_1S_1, L_2S_2, etc., because in the vessel's actual inclined position the water-line would in each case be horizontal, while the line of action is vertical. Thus, for the extreme angle of inclination shown in the figure, the line B_3m_3 at right angles to L_3S_3 would be the line of action of the buoyant force; and the point m_3, where this line cuts the centre line of the vessel, will evidently be a critical point; for if the centre of gravity G lies below m_3, the forces will exert a righting moment, but if G lies above m_3, the moment will be an upsetting moment, and the vessel will not return towards the upright position, but will heel still further towards the right.

The exact height of the point m_3 can readily be calculated, for if we draw B_3K_3 parallel to the floor, the triangle $B_3K_3m_3$ will be similar to L_3L_0C, and its height will be—

$$\mathrm{K}_3m_3 = x \times \frac{b}{\delta} = \frac{b\delta}{3D} \times \frac{b}{\delta} = \frac{b_2}{3D} \quad . \quad . \quad . \quad . \quad (1)$$

If we proceed in the same way from either of the intervening points B_2 or B_1, it is evident that we shall find in each case the same height of triangle, or generally $Km = \frac{b^2}{3D}$, the length of the subnormal being a constant quantity in the parabolic curve.

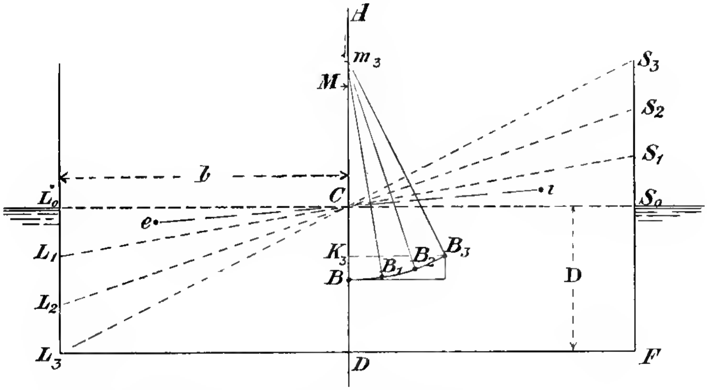
The point K at the base of the triangle KBm, is, however, not a fixed point in the vessel, but lies above the known point B_0 by a vertical height $KB_0 = y = \frac{\delta_2}{6D}$. Thus the height B_0m is not constant, but is equal to a constant + the variable quantity y, or—

$$B_0m = \frac{b}{3D} + \frac{\delta^2}{6D} \quad . \qquad . \qquad (2)$$

In the upright position of the vessel, the moving point m reaches the limiting position M_0, and as y is now nothing, the height of M_0 above the centre of buoyancy is—

$$B_0M_0 = \frac{b^2}{3D}, \text{ or } = \tfrac{2}{3}\frac{b^3}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

FIG. 39



The point M_0 is the proper "metacentre" of the pontoon, and its position relatively to G determines the question of stability *in the upright position.* Here stability can only be obtained by so loading or ballasting the vessel as to bring the centre of gravity G below M_0, and if G is at any level higher than M_0, the vessel cannot float upright.

Strictly speaking, we cannot quite say that if G is above M_0 the pontoon will roll completely over. If G happened to be situated at the point m_2, for example, the vessel could not float upright, and must cant over to the right or to the left; but on reaching the position for which L_2S_2 is the water-line, she would find an equilibrium that is nominally stable, although practically it might be highly perilous.

EXAMPLE.—Let the pontoon have a moulded width of 24 feet, and a draught of 6 feet, the point B_0 being consequently 3 feet below the load water-line.

To trace the curve of buoyancy up to the angle L_0CL_3, we may divide the height L_0L_3 into three equal parts, and for the three positions of B we shall have the following co-ordinates :—

$$\begin{aligned}
\delta_1 &= 2;\ x_1 = \frac{b\delta}{3\text{D}} = \frac{12}{3\times6}\delta = 1{\cdot}33;\ y_1 = \frac{\delta^2}{6\text{D}} = \frac{\delta^2}{36} = \tfrac{4}{36} = 0{\cdot}11\\
\delta_2 &= 4;\ x_2 = \quad\text{,,}\quad = 2{\cdot}67;\ y_2 = \quad\text{,,}\quad = \tfrac{16}{36} = 0{\cdot}44\\
\delta_3 &= 6;\ x_3 = \quad\text{,,}\quad = 4{\cdot}00;\ y_3 = \quad\text{,,}\quad = \tfrac{36}{36} = 1{\cdot}00
\end{aligned}$$

The height of the metacentre M_0 above the centre of buoyancy will be $\frac{b}{3\text{D}} = \frac{12^2}{3\times6} = 8{\cdot}00$ feet, its height above the water-line being therefore 5 feet. But as the pontoon rolls to the successive angles, the height of the point m increases as y increases, so that B_0m comes to 8·11, 8·44, and finally to 9·00 feet.

Pontoons of rectangular form have often been employed in the erection of large girder bridges for the purpose of transporting the girders to the site; and sometimes, as in the case of the Hawkesbury bridge, the girders have been carried at a great height above the deck, with the object of placing them upon the piers without lifting.

For any such purpose, it is obviously necessary to design the pontoon with a great height of metacentre, and the object can most easily be attained by adopting a liberal width and a shallow draught.

In the example worked out above, the immersed section had an area 24 feet × 6 feet = 144 square feet; and without altering the displacement of the pontoon or the area of the rectangle, a much greater metacentric height might easily be obtained by changing its proportions; thus, by choosing—

		A.	B_0M_0.	CM_0.
(1)	A half-breadth $b=12$, D$=6$, we have	144 square feet	8·0	5·0
(2)	,, $b=18$, D$=4$, ,,	144 ,,	27·0	25·0
(3)	,, $b=24$, D$=3$, ,,	144 ,,	64·0	62·5

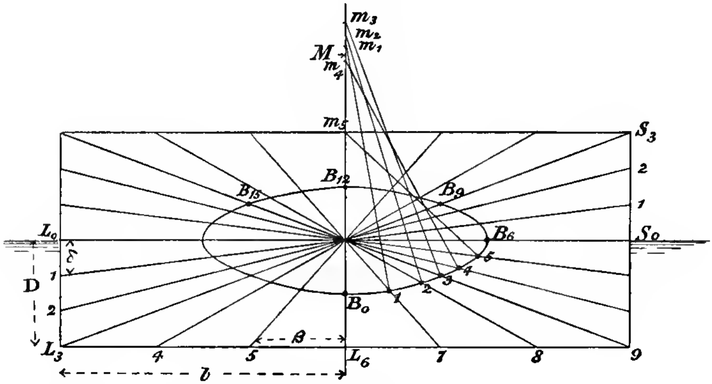
If, however, we choose a very great width and a very shallow depth, it will follow that a very small angle of heel will suffice to plunge the deck under water at one side, or to lift the floor out of water on the other side. To study the practical limitations of choice in this direction, it will be necessary to extend the construction of the curve of buoyancy.

Art. 49. Curve of Buoyancy for Rectangular Vessels.—The construction of the complete curve of buoyancy, although rather a tedious matter in shipbuilding, is simple enough in the case of rectangular pontoons.

Taking, first, the case shown in Fig. 40, where the height of freeboard is just equal to the draught, we may draw the water-lines for all angles of inclination through the central point C; and assuming the deck to be water-tight the curve can be traced for a complete revolution.

In the figure the water-lines are drawn in twelve positions (for twenty-four positions of the vessel), each side of the rectangle being divided into six equal parts. It has already been shown that, up to position No. 3, the curve is a parabola starting at B_0 and terminating at B_3 in the centre of gravity of the triangle $L_3 9 S_3$, the co-ordinates being easily calculated from the variable depth δ. In precisely the same way it may be seen that from B_3 to B_6 the curve will be another parabola having the line CB_6 for its axis, the co-ordinates being now calculated from the variable quantity β measured from the centre of the floor, and having the values $\frac{D\beta}{3b}$ and $\frac{\beta^2}{6b}$. Thus the entire curve consists of four parabolic arcs which meet upon the diagonals of the rectangle, or upon the four corners of an inner rectangle, the base $B_9 B_{15}$ being one-third of the total width of the pontoon, while the base $B_3 B_9$ is one-third of its height. The distance CB_6 is, of course, one-fourth of the total width, and CB_0 one-fourth of the height.

FIG. 40



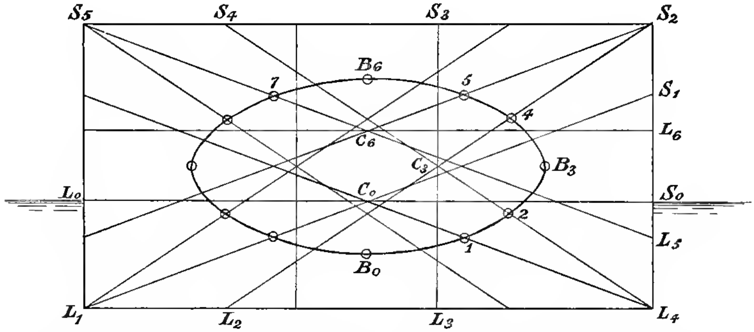
The normals Bm are drawn, in the figure, through the points that mark the successive positions of B, and at right angles to the corresponding water-lines. These normals show that after passing the third position, the point m falls rapidly to lower positions $m_4 m_5$ on the centre-line, and finally to the centre C.

The curve of buoyancy becomes slightly altered when, as in Fig. 41, the draught L_0L_1 is not equal to the freeboard L_0S_5.

Beginning with the water-lines and starting from the original position L_0S_0, we may divide the height L_0L_1 into any convenient number of parts; and, as before, all the water-lines will pass through the point C_0, until the position $L_1C_0S_1$ is reached, and now the point B_1 will be at the centre of gravity of the triangle $L_1L_4S_1$. The curve B_0B_1 will therefore be a parabola constructed in the manner already described.

As the vessel heels over still further, the immersed section will maintain a triangular form until the position L_2S_2 is reached; and the triangle $L_2L_4S_2$ must obviously have the same area as the triangle $L_1L_4S_1$, so that we have for its base $L_2L_4 = L_1L_4 \times L_4S_1 \div L_4S_2$. The point B_2 will now be at the centre of gravity of the triangle $L_2L_4S_2$.

FIG. 41



When the vessel has rolled through 90°, the immersed section will again be a rectangle, having the same area as before, so that the water-line L_3S_3 bisects the base L_2L_4 of the last-named triangle, and the point B_3 will now be in the centre of the immersed rectangle. The curve B_2B_3 will again be a parabola, and may be traced as before. Thus, dividing L_2L_3 into any convenient number of parts, and denoting by β the distance of either of these points of division from L_3, the co-ordinates of the curve will be $\frac{\mathrm{H}\beta}{3\mathrm{L}}$ and $\frac{\beta^2}{6\mathrm{L}}$, H being the total height of side L_4S_2, and L being the length L_2L_3.

The curve B_1B_2, between the parabolic arcs B_0B_1 and B_2B_3, will be a hyperbola, having for its asymptotes the sides of the rectangle L_1L_4 and L_4S_2. Thus the entire curve consists of four parabolic and four hyperbolic curves. The construction presents no difficulty, but each individual case must be treated according to its special dimensions.

Art. 50. Diagram of Righting Moments.—Having traced the curve of buoyancy and drawn the normals Bm through the successive positions of B, the righting moment exerted by the forces W and B can at once be measured for either of the inclined positions as soon as we know the true position of the centre of gravity G.

Thus, in Fig. 40a, if we draw GN at right angles to B_3m_3, the righting moment for the angle of heel θ_3 will be $\mathrm{W} \times GN = \mathrm{W} \times Gm_3 \,.\, \sin.\, \theta_3$.

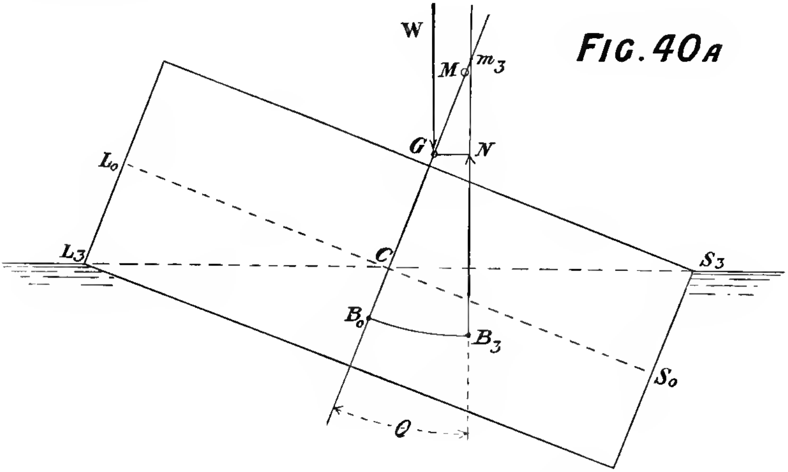


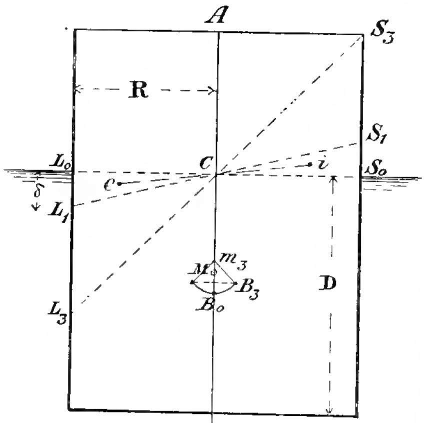
FIG. 40A

Then plotting as abscissæ the successive angles of roll θ_1, θ_2, etc., we can erect co-ordinates to represent the varying values of the righting moment. The form of this diagram would depend upon the proportions of the pontoon and also upon the height of the centre of gravity; and in different cases it would assume a variety of widely different shapes. Its chief practical use would be to determine the value of two critical angles of roll—first, the angle θ_m, at which the righting moment attains its maximum; and second, the angle θ_0, at which the righting moment vanishes. For a vessel sailing under press of canvas, the angle θ_m would be the greatest angle of heel consistent with safety; while for a pontoon rolling after the effects of a swell, the angle θ_0 would be the extreme limit, which could not be overpassed without causing the vessel to capsize.

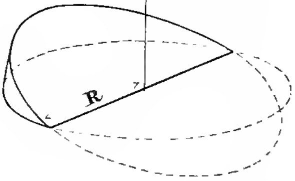
These angles can be found by inspection of a cage diagram, such as Fig. 40, almost as well as upon a diagram of righting moments. Thus, if we suppose the centre of gravity to be situated just above the deck, at the point m_5, it will at once be seen that the angle of greatest righting moment θ_m is either the angle $B_0m_3B_3$, or an angle approximating very closely thereto; while the limiting angle θ_0 is obviously the angle $B_0m_5B_5$.

In rectangular pontoons of different proportions, the angles θ_m and θ_0 may have widely different values, and will depend, not only on the proportions of the rectangle, but also on the position of the centre of gravity. Each case requires, in fact, to be individually examined.

FIG. 42



42A



Art. 51. Cylindrical Vessel floating on End.—In Fig. 42 the vertical line ACB_0 is intended to represent the axis of a cylindrical caisson or other vessel having a flat bottom, and floating in an upright position. If R denotes the radius of the cylinder, and D the vertical depth of the flat bottom below the water-line, the displacement will be πDR^2; so that if the weight W is known, we shall have $D = \frac{35W}{\pi R^2}$ (for salt water).

In whatever direction the vessel may roll, the wedges of emersion and immersion will have the form sketched in Fig. 42*a*; and the volume of either wedge will be $\frac{2}{3}\delta R^2$, δ being the extreme height of the wedge measured vertically upon the side of the cylinder as in Fig. 42. The ratio of this volume to the total displacement will therefore be expressed by $\frac{2\delta}{3\pi D}$; and by the method already referred to in Art. 48, we can easily trace the curve of buoyancy for all inclinations up to the limiting position of the water-line L_3S_3, at which the water-mark reaches either the upper or the lower edge of the immersed cylindrical body.

For in Fig. 42 the centre of gravity e of the wedge of emersion lies to the left of the centre-line at a known distance of $\frac{3}{16}\pi R$, measured parallel to S_0L_0, and below the line S_0L_0 at a distance of $\frac{3}{32}\pi\delta$, measured parallel to AC. The co-ordinates for the curve of buoyancy will therefore be—

$$x = 2 \times \tfrac{3}{16}\pi R \times \frac{2\delta}{3\pi D} = \frac{\delta R}{4D}$$

$$\text{and } y = 2 \times \tfrac{3}{32}\pi\delta \times \frac{2\delta}{3\pi D} = \frac{\delta^2}{8D}$$

The curve is therefore again a parabola, having the centre-line ACB_0 for its axis, and the point B_0 for its vertex, midway between the water-line and the floor.

The critical point m, where a normal through the new centre of buoyancy cuts the centre-line, may also be found in the manner already described, its height above B_0 being expressed by—

$$B_0m = x\frac{R}{\delta} + y = \frac{R^2}{4D} + \frac{\delta^2}{8D} \quad . \quad . \quad . \quad . \quad (4)$$

For the upright position the point of intersection becomes the metacentre M_0, and as $y = 0$, we have—

$$B_0M_0 = \frac{R^2}{4D} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be seen that if the centre of gravity G were exactly at the centre of the cylinder's total height, the vessel could only float upright on condition that the height of freeboard S_0S_3 is less than twice the height B_0M_0, or in other words, less than $\frac{R^2}{2D}$; but in many cases the centre of gravity may be *lower* than halfway down from the upper edge to the floor.

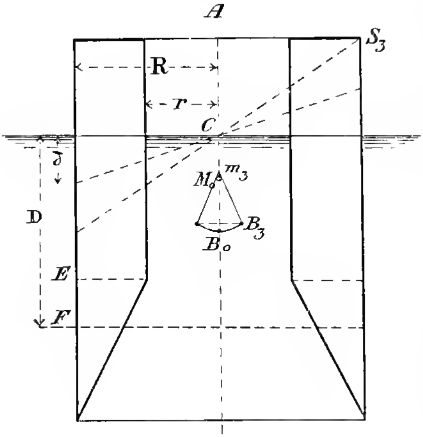
Art. 52. Annular Vessel, or Double-skinned Cylindrical Caisson.—Let Fig. 43 represent the section of a caisson of cylindrical form, with an inner concentric wall of thin plate, forming a central chamber or dredging well; and leaving between the walls a closed annular space whose buoyancy is used, as we will suppose, to float the caisson.

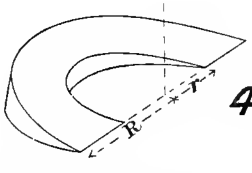
The horizontal section of the annular chamber will have the area $\pi(R^2 - r^2)$, in which R is the radius of the outer and r of the inner shell. The displacement is therefore easily to be found; but if the base of the caisson is formed with a cutting edge, the cubic capacity of the space contained between the outer cylinder and the inner cone will have to be separately calculated. Then, dividing that calculated volume by the general sectional area $\pi(R^2 - r^2)$, we may obtain an equivalent height $\overline{EF}$, which may be set off below the shoulder E, and the effective draught D, measured down to this point, will then give us the total displacement, or $V = D\pi(R^2 - r^2)$. Then, if the weight W is known beforehand, we shall have (for salt water) $V = 35W$, or—

$$D = \frac{35W}{\pi(R^2 - r^2)}. \quad (6)$$

In whatever direction the floating caisson may roll, the wedges of emersion and immersion will have the form sketched in Fig. 43a; and the volume of either wedge will be $\frac{2}{3}\frac{\delta}{R}(R^3 - r^3)$, in which δ is the extreme height of the wedge measured vertically upon the outer side of the cylinder, as indicated in Fig. 43.

FIG. 43





43A

The ratio of this volume to the total displacement will therefore be $\frac{2\delta(R^3 - r^3)}{3R\pi D(R^2 - r^2)}$.

The centre of gravity of the wedge of emersion lies to the left of the centre-line, at the distance—

$$\frac{3\pi}{16} \cdot \frac{R^4 - r^4}{R^3 - r^3} \text{ (measured parallel to the base)}$$

and below the water line at the distance—

$$\frac{3\pi}{32} \cdot \frac{\delta}{R} \cdot \frac{R^4 - r^4}{R^3 - r^3} \text{ (measured parallel to } AC\text{)}$$

The co-ordinates of the curve of buoyancy are therefore—

$$x = \frac{\delta}{4RD} \cdot \frac{R^4 - r^4}{R^2 - r^2}, \text{ and } y = \frac{\delta^2}{8DR^2} \cdot \frac{R^4 - r^4}{R^2 - r^2}$$

It follows, therefore, that the curve is again parabolic, having ACB_0 for its axis, and for its vertex the point B_0, which marks the position of the centre of buoyancy for the upright position of the caisson.

For any value of δ less than the total height of freeboard S_0S_3, the point of intersection m, which has already been referred to, will be situated at a height above B_0, expressed by—

$$\overline{B_0m} = x\frac{R}{\delta} + y = \frac{1}{4D}\left(\frac{R^4 - r^4}{R^2 - r^2}\right)\left(1 + \frac{\delta^2}{R^2}\right) \quad . \quad (7)$$

and for the upright position of the caisson, when $y = 0$, the metacentric height will have the value—

$$\overline{B_0M_0} = \frac{1}{4D} \cdot \frac{R^4 - r^4}{R^2 - r^2} \quad . \quad . \quad . \quad . \quad (8)$$

Art. 53. Negative Metacentric Height.—We have seen that the stability of a vessel, in its upright position, depends upon the position of its centre of gravity relatively to a certain point, which may be regarded as an instantaneous point of suspension. In the examples referred to in Arts. 44, 45, and 46, this point was found to be the centre of buoyancy; while in all the cases that have since been examined in Arts. 47 to 52, the metacentre was found to be situated at a certain height *above* the centre of buoyancy, so that the vessel might still possess stability although her centre of gravity were higher than the centre of buoyancy, so long as it was still some distance below the metacentre. On the other hand, it is easy to see that the opposite condition of things may sometimes arise, and that the metacentre, if we can still continue to call it by that name, may be at some distance *below* the centre of buoyancy instead of being above it.

Thus, for example, let Fig. 44 represent the section of a rectangular tank partly filled with water; and let us imagine this vessel to be wholly submerged. We have already seen that, as regards motion in a vertical direction, such a vessel would be in a condition of indifferent equilibrium, and would not be likely to remain at one level for any length of time unless it touched the bottom. But, confining our attention just now to the rolling motion, it is obvious that when the tank rolls over to the right, as sketched in Fig. 44*a*, the effective centre of buoyancy will move away towards the *left* instead of towards the right; and in the new position B_1, the line of action of the supporting force will be the vertical line m_1B_1.

There will be no righting moment, therefore, unless the centre of gravity of the symmetrically built tank (exclusive of contained water) lies below the point m_1.

FIG. 44

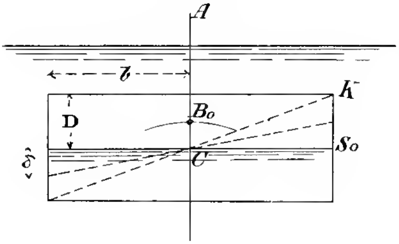
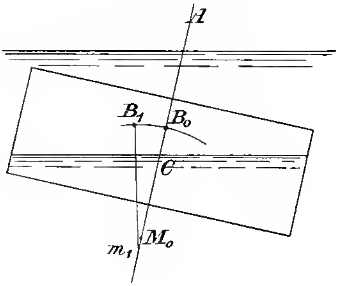


FIG. 44A



A little consideration will show that the negative height B_0M_0 to the metacentre, or B_0m to any point of intersection m, can be found by the same method that we have used for determining those heights when they were positive quantities.

At every point the vertical buoyant pressure per square inch of horizontal section will be proportional to the vertical height from the surface of the contained water to the roof of the tank; and this will be equally true whether the tank is closed on all sides or made like a diving-bell with an open bottom.

The centre of buoyancy will, therefore, always coincide with the centre of gravity of the air-space (neglecting both the buoyancy and the weight of the water-filled space); and as the water rolls, the figure of the air-space will undergo the same changes as they would do if we turned the diagram upside down and regarded the interior water-line as though it were the exterior water-line of a floating rectangular pontoon.

The curve of buoyancy will, therefore, be a parabolic curve, whose co-ordinates will be expressed by the formulæ already given in Art. 48, but this time the curve will be convex upwards. The arc will continue to be parabolic until the inside water surface reaches one of the corners of the tank; and if we wish to follow the curve farther, we can adopt the methods already described in Art. 49. The negative metacentric height B_0M_0 for the upright position will, therefore, be given by the formula that was worked out for the rectangular floating pontoon; *i.e.*—

$$-B_0M_0 = \frac{b^2}{3D} \quad . \quad . \quad . \quad (9)$$

in which b is the half-breadth as before, and D is now the height of air-space KS_0.

For any angle of roll in which δ is not greater than KS_0, we have the general formula—

$$-B_0m = \frac{b^2}{3D} + \frac{\delta^2}{6D} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To obtain stability in the upright position, it is obvious that the centre of gravity of the shell must lie below M_0; and in the case of the flat tank sketched in the figure, if the centre of gravity were higher than M_0 (as it would be generally), the tank must roll over until it swims edgewise in the water. If the centre of gravity were brought (by external loading) down to a point *a little* lower than M_0, say to m_1, the tank might swim upright and possess a slight stability within a limited angle of roll; but at that limit the stability would vanish.

It is also evident that the greater the proportion of b to D, the greater will be the negative metacentric height, or the *instability* (shall we say) of the submerged vessel. Thus, the stability of ships is greatest in the longitudinal direction, the metacentric height for longitudinal pitching and scending being very great; and for the same reason the longitudinal *instability* is very great in the submerged vessel; so that when a ship goes down with any contained air-space, she is almost certain to sink end foremost.

The method described in this article will, of course, apply to other forms of submerged vessel partly filled with water. Thus, for a submerged cylinder with a vertical axis, we should have—

$$-B_0M_0 = \frac{R^2}{4D}; \text{ and } -B_0m = \frac{R^2}{4D} + \frac{\delta^2}{8D}$$

as found in Art. 51.

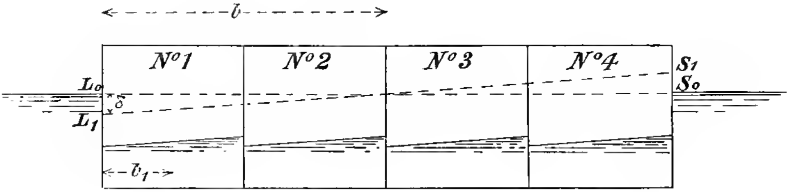
Art. 54. Effect of Water-ballast in Compartments.—The pontoons employed in the erection of girder bridges have sometimes been built in compartments, with the intention of admitting a certain amount of water-ballast, so as to bring the pontoon to a deep draught or to raise it again through a certain height by pumping out the whole or a part of the water.

When the compartments are partly filled with water, the stability of the pontoon will be somewhat diminished, and the metacentric height B_0M_0 will be less than the quantity $\frac{b^2}{3D}$, which was found in Art. 48 for a pontoon of the same width and displacement. But the effect of the water-ballast in reducing the metacentric height will be great or small according to the width of the individual tanks, or the number of compartments contained in the whole width of the pontoon.

To illustrate the question, we will suppose the pontoon to be divided, transversely, into a certain number N of equal compartments, the tanks extending throughout the whole length of the pontoon, and all having the same width between bulkheads, as sketched in the cross-section, Fig. 45, in which N = 4.

If all the tanks are filled to the same level, and if D denotes now the vertical height of the exterior water-line above the water in the tanks, the net displacement of the pontoon (apart from water-load) will be equal to the length multiplied by 2*b*D.

FIG. 45



When the vessel rolls so as to bring the exterior water-mark to the inclined position S_1L_1, the water surface in each tank will be parallel to S_1L_1; and the figure of the area of displacement will be changed by the transference of a parallelogram from No. 1 tank to No. 4; and of another parallelogram from No. 2 to No. 3.

Taking in succession the computed area of each parallelogram multiplied by the distance through which it is transferred (horizontally and vertically), and dividing the sum by the displacement area 2*b*D, we have the co-ordinates of the curve of buoyancy, viz.—

$$x = \frac{2\delta b}{\mathrm{DN}^3}\{(\mathrm{N}-1)^2+(\mathrm{N}-3)^2+\text{etc.}\} = \frac{\delta b}{3\mathrm{D}}\left(1-\frac{1}{\mathrm{N}^2}\right) \quad . \quad (11)$$

$$\text{and similarly, } y = \frac{\delta^2}{6\mathrm{D}}\left(1-\frac{1}{\mathrm{N}^2}\right) \quad . \quad . \quad . \quad . \quad (12)$$

So long as the water plays up and down upon vertical walls and bulkheads without reaching the exterior corners of the pontoon or the interior corners of the tanks, the curve of buoyancy will be parabolic; and for any value of δ within these limits the corresponding height B_0m will be—

$$\overline{\mathrm{B}_0m} = x\frac{b}{\delta} + y = \left(1-\frac{1}{\mathrm{N}^2}\right)\left(\frac{b^2}{3\mathrm{D}}+\frac{\delta^2}{6\mathrm{D}}\right) \quad . \quad (13)$$

For the upright position, where $y = 0$, we have the metacentric height—

$$\mathrm{B}_0\mathrm{M}_0 = \frac{b^2}{3\mathrm{D}}\left(1-\frac{1}{\mathrm{N}^2}\right) \quad . \quad (14)$$

It is then easy to see the way in which the metacentric height decreases, as the number of compartments is decreased, thus—

If $\mathrm{N} = 1$, the height $\mathrm{B}_0\mathrm{M}_0 = 0$
$\mathrm{N} = 2$, „ $\mathrm{B}_0\mathrm{M}_0 = \frac{b^2}{3\mathrm{D}} \times \frac{3}{4}$
$\mathrm{N} = 3$, „ $\mathrm{B}_0\mathrm{M}_0 = \frac{b^2}{3\mathrm{D}} \times \frac{8}{9}$
$\mathrm{N} = 4$, „ $\mathrm{B}_0\mathrm{M}_0 = \frac{b^2}{3\mathrm{D}} \times \frac{15}{16}$

The quantity $\frac{b^2}{3\mathrm{D}}$ is, of course, the metacentric height for the same pontoon, carrying the same external load, and having the same net displacement, but with the compartments pumped dry.

The negative quantity $\frac{b^2}{3\mathrm{D}}\left(\frac{1}{\mathrm{N}^2}\right)$ has the same value as the negative metacentric height for each individual tank whose half-breath is $b_1 = \frac{b}{N}$, and might be written $\frac{b_1^2}{3D}$, in which case the metacentric height of the pontoon would be—

$$B_0M_0 = \frac{b^2 - b_1^2}{3D} \quad . \quad . \qquad (15)$$

If all the tanks are partly filled, but some higher than others, the net displacement for the same external load will still remain the same, and we may still use the above expressions if D is understood to denote the *mean* depth of the contained water below the external water-level.

In this article we have considered only the stability of the pontoon in the transverse plane, and have assumed the tanks to extend throughout the whole length. But the subdivision of the pontoon longitudinally by transverse bulkheads, and the stability in the longitudinal plane, may be treated in the same manner.

CHAPTER IX.

BENDING STRESSES IN FLOATING PONTOONS.

Art. 55. Distribution of the Vertical Forces.—The bending stress that takes effect in any floating vessel will, of course, depend wholly upon the distribution of the load and the distribution of the supporting buoyant pressure. In shipbuilding it is mainly the longitudinal stresses that have to be considered; and the principal bending moments are the hogging moment due to the inadequacy of the buoyant support at the ends of the vessel, and the contrary sagging moment which sometimes takes place when the centre of the ship's length lies in the trough of the sea.

When the ship floats in calm water, it is easy to see that, if the weight of ship and cargo per lineal foot were everywhere proportional to the area of the immersed section, there would be no longitudinal bending stress whatever. This equable distribution of the opposing forces is not quite attained in practice, and in calm water the ship is generally subjected to a hogging stress owing to the fineness of the water-lines at the ends. But in pontoons that are built for the support of heavy concentrated loads, the distribution of the opposing forces is often such as to produce a much greater bending stress (in proportion to the total load), and the bending may take effect not only in the longitudinal but in the transverse plane.

Sometimes the problem would be very unmanageable were it not for the fact that, when the water is calm, we know, almost exactly, how the supporting buoyant pressure must be distributed.

For, whatever may be the form of a pontoon, we have already seen that the vertical buoyant pressure per square inch of horizontal section is everywhere measured by the vertical depth of the floor below the water-line, so that a section of the vessel, with the water-line drawn across it as a datum, gives us at once a diagram of the distributed buoyant force.

We need only consider the case of a flat-bottomed rectangular pontoon, and it is obvious that if the load is symmetrically trimmed so as to bring the vessel upright, *i.e.* if the floor is a horizontal plane, the buoyant pressure will be a uniformly distributed force as in Fig. 36; and if the centre of gravity does *not* coincide with the centre-line of the pontoon, the diagram will have the very simple form sketched in Fig. 36*a*, and the actual distribution can easily be found.

The calculation will, therefore, be very simple when we know the disposition of the load; but it is here that a difficulty sometimes arises, because the actual pressure of the external load at any given point is often affected by the elastic deflection of the pontoon itself under the bending moment. Another complexity arises when the deflection of the pontoon is great enough (as it sometimes may be) to sensibly affect the figure of the area of displacement, and therefore the distribution of the buoyant pressure.

We may assume that the weight of the pontoon itself will generally have a nearly constant value per square foot of floor; and as it will be borne by a uniformly distributed water pressure, these opposite forces may be eliminated from both sides of the account, as they will produce no bending moment under any circumstances. What we have to consider, therefore, is the external load, opposed by a buoyant force of equal magnitude, and the moments produced by these opposite forces.

Art. 56. General Expressions for the Bending Moment and the Vertical Shearing Force.—The question we have in hand involves only a somewhat special application of well-known principles. We may consider the pontoon as a girder of uniform depth, subjected to the action of certain vertical forces which we may treat as positive or negative according to the upward or downward direction in which they act, taking the buoyant pressures as positive and the loads as negative. Then, at any vertical section taken through the girder, the shearing force S will be simply the algebraical sum of all those forces which act anywhere to the *right* of the section, while the bending moment M will be the algebraical sum of the moments of those forces.

Thus if $\pm$ P denotes any vertical pressure, upward or downward, whose line of action or centre of action lies to the right of the section by a horizontal distance X, its moment at the section in question is ± PX; and taking all such forces in succession, we have—

$$S = \Sigma(P) \text{ and } M = \Sigma\,.\,(PX).$$

In these general expressions the negative forces $(-P)$ are loads either distributed or concentrated at given points; but if we neglect the slight alteration of form due to bending, the positive force is always a distributed upward pressure, whose uniform intensity p is easily to be found from the known loads.

Art. 57. Pontoon carrying a Central Load.—We will suppose the pontoon sketched in Fig. 46 to be loaded with the weight of a heavy girder, or of a deep and heavy ship, lying centrally along the whole length of the pontoon, whose cross-section is here represented, and that the load is supported at certain intervals by keel-blocks, which are placed immediately over the cross-girders that extend transversely between the floor-plates.

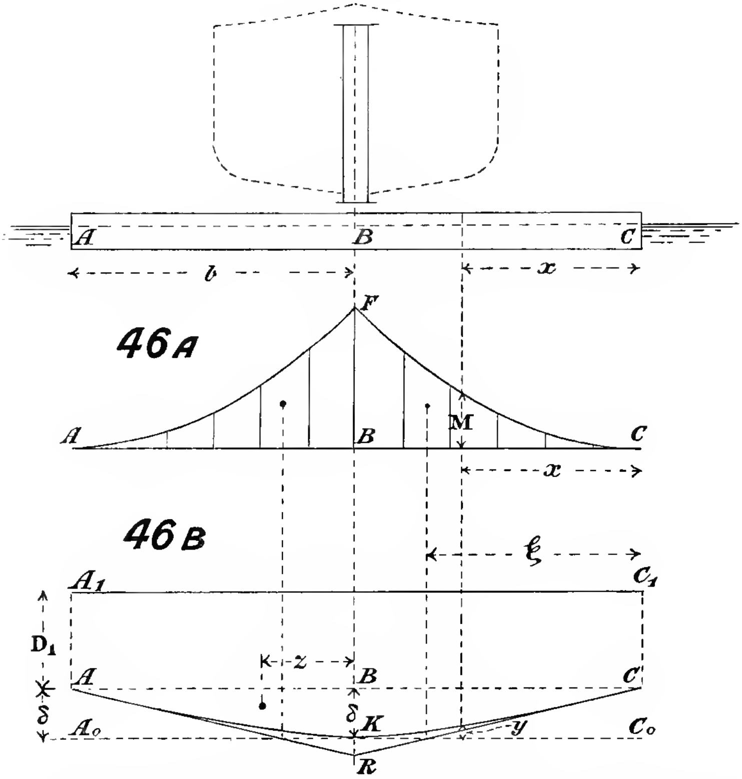
Apart from the weight of the pontoon itself, and apart from the uniform buoyant pressure which carries it, we have to consider the external load consisting only of the weight of the carried ship or railway girder; and, in the first instance, we will assume that the rigidity of the ship or girder is sufficient to prevent any bending of the pontoon in the longitudinal plane, so that the draught of water is the same at each cross-girder of the pontoon. Under these conditions we might reckon without much error that, if the cross-girders divide the length of the pontoon into N equal bays, and if the keel-blocks are all adjusted at the same height, the load on each will be $\frac{1}{N}$th of the total weight W of the ship or girder; while the mean buoyant pressure per lineal foot of cross-girder will be $p = \frac{W}{2bN}$, in which b is the half-width of the pontoon. Or, if w is the external load per lineal foot of the pontoon's *length*, and s the length of each bay, we should have $p = \frac{ws}{2b}$.

For a first approximation we may neglect also any bending of the pontoon in the transverse plane; and the draught being uniform in both planes, the distributed force p would be uniform all along the cross-girder.

Then, taking a vertical section at any point, whose transverse distance from the nearest edge of the pontoon is x, we have generally $\mathrm{M} = \frac{px^2}{2}$, and $\mathrm{S} = px$.

The greatest bending stress will be at the centre B, where $\mathrm{M_B} = \frac{pb^2}{2}$, and the diagram of moments will consist of a pair of parabolic curves, whose vertices are at the ends of the pontoon A and C, as shown in Fig. 46a.

FIG. 46



The greatest shearing force will take place immediately to the right or left of the centre, where it will have the value $\mathrm{S_B} = pb = \frac{\mathrm{W}}{2\mathrm{N}}$, and from the centre it will diminish regularly to the ends of the cross-girder, where it will be zero.

In practice the transverse bending of a pontoon would generally be too small to produce any considerable change in the figure of the area of displacement; but in the case of a long pontoon loaded in the middle of its *length*, the deflection in the longitudinal plane might be very considerable; and if we may regard such a pontoon as a girder of uniform depth and uniform section, it will not be difficult to trace the curve of deflection.

Under the stresses due to the moments already calculated, it would, in fact, bend into a curve, which is the quartic parabola drawn in Fig. 46*b*, to an exaggerated vertical scale.

Taking either half of the curve to the right or left of the central point K, the two tangents KC_0 and CR must intersect each other at a point directly below the centre of gravity of the area FBC in the diagram of moments. The inclination of the tangent CR will be proportional to the same area FBC, and equal to that area divided by the product EI, in which E is the modulus of elasticity and I the moment of inertia of the pontoon's cross-section. The deflection CC_0, which is the same thing as BK, will be proportional to the moment of the same area BFC about the point C, and equal to that moment divided by EI.[1]

In the present case we have for the area BFC the value $\mu = \int_0^b \mathrm{M}dx = \frac{pb^3}{6}$; and for its moment about the point C the value $\int_0^b \mathrm{M}xdx = \frac{pb^4}{8}$; so that its centre of gravity lies at a distance $\xi = \frac{3}{4}b$ from the end C.

Therefore, in the deflection curve of Fig. 46*b*, the inclination of the tangent CR will be $i = \frac{pb^3}{6\mathrm{EI}}$; so that $BR = ib = \frac{pb^4}{6\mathrm{EI}}$; and it is obvious that $\overline{BK} = 3\overline{KR} = \frac{3}{4}\overline{BR}$, whence it follows that $\overline{BK} = \frac{pb^4}{8\mathrm{EI}}$. Or, by the general rule, we find the deflection—

$$CC_0 = \delta = \frac{\mu\xi}{\mathrm{EI}} = \frac{pb^4}{8\mathrm{EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the pontoon has no external stiffening, we can express more simply the bending moment and the central deflection in terms of the uniform depth d, and the uniform section a of either flange. The maximum stress per square inch (at the centre B) will then be $f_B = \frac{pb^2}{2ad}$; and if we regard the diagram of Fig. 46a as a diagram of stress intensity, in which the maximum stress f_B is represented by the ordinate FB, we may calculate the deflection by the formula—

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[1] The graphic method of tracing the curve of deflection from any given diagram of moments is described in a paper on "Continuous Girder Bridges," published in vol. lxxiv. of the *Transactions of the Inst. C.E.*; and in greater detail in "A Treatise on Bridge Construction."

$$\delta = f_B \cdot \frac{b^2}{2\mathrm{E}d} = \frac{pb^4}{4\mathrm{E}ad^2} \quad . \quad . \qquad . \quad . \quad (1a)$$

Hence the ratio $\frac{\delta}{b}$ is proportional to the ratio $\frac{b}{d}$, and to the quantity $\frac{f_B}{2\mathrm{E}}$, in which we may take E at the value of about 13,000 tons per square inch for steel plate; and if we assume that f_B will not exceed an ordinary value for the working stress, we can readily estimate the probable limits to this elastic deflection.

Thus, if we assume for f_B a maximum stress of $4\frac{1}{3}$ tons per square inch of gross section, the quantity $\frac{f_B}{2\mathrm{E}}$ will be $\frac{4\frac{1}{3}}{26{,}000}$ $= \frac{1}{6000}$; and we shall then have $\delta = \frac{b}{6000} \times \frac{b}{d}$.

For example, if the pontoon were 90 feet wide and 6 feet deep, a central load producing a stress of $4\frac{1}{3}$ tons per square inch would produce a deflection of $\frac{45}{6000} \times \frac{45}{6} = 0{\cdot}056$ foot, or a little more than half an inch.

But if the pontoon were 360 feet long, with the same depth, and were designed to carry a central load with the same working stress, the deflection in the longitudinal plane would be very much greater, and would amount to $\frac{180}{6000} \times \frac{180}{6} = 0{\cdot}90$ foot, or about $10\frac{3}{4}$ inches. If the mean draught of such a pontoon were not more than about 3 feet, it is obvious that a deflection of 10 or 11 inches would sensibly alter the distribution of the buoyant pressure.

To complete the deflection curve of Fig. 46b, the simplest plan is to set off vertical ordinates above the tangents CR and AR, whose height at any point will be proportional to x^4, and will be expressed by $y = \frac{px^4}{24\mathrm{EI}}$.

Subtracting from the triangular area CBR the area $\int y\,dx$, we have for the submerged area CBK the value $\frac{3}{5}b\delta$.

Subtracting from the moment of the triangular area the moment $\int yx\,dx$, and dividing by $\frac{3}{5}\delta b$, we find that the centre of gravity of the submerged area *CBK* lies at a distance of $\frac{35}{54}b$ from the point *C*, or at a distance from *B* which is expressed by $z = \frac{19}{54}b$; so that the moment of that area about *B* will be expressed by $\frac{19}{90}\delta b^2$.

Having thus found the elements of the curve that has been drawn in Fig. 46*b*, we can go on to inquire how the stresses would be affected by such a change in the figure of the immersed section, which would now consist of a rectangle bD_1 and the submerged area *ABK*, for each half of the pontoon. If D is the mean draught, or the uniform draught of an *inflexible* pontoon, it is obvious that we have $D = D_1 + \frac{3}{5}\delta$, inasmuch as the immersed section must have the same area whether the bending be much or little; so that $D_1 = D - \frac{3}{5}\delta$, while at the centre the draught would be $A_1A_0 = D + \frac{2}{5}\delta$. Regarding *AC* as the *length* of the pontoon, we can simplify matters by considering the load w imposed (at *B*) upon each lineal foot of its width, and writing in terms of w, the forces and stresses for that one foot of width, we have for the straight pontoon—

$$M_B = \frac{pb^2}{2} = \frac{wb}{4} \quad . \quad . \quad . \qquad . \quad (2)$$

or generally—

$$M = \frac{wx^2}{4b} \quad . \quad . \quad . \qquad . \quad (3)$$

and under this bending stress—

$$\delta = \frac{wb^3}{8Ead^2} \quad . \qquad . \quad . \quad . \quad . \quad (4)$$

If, therefore, the pontoon were built with an initial camber equal to this calculated value of δ, and to the form of Fig. 46*b* inverted, it would come to a straight line under the central load w, and the bending moment would then have the value above given.

On the other hand, if the floating pontoon has the curved form of Fig. 46*b*, the mean draught being $D = \frac{w}{2b\gamma}$, the moment at *B* due to the rectangle D_1b will be $\gamma(D - \frac{3}{5}\delta)\frac{b^2}{2}$, while that due to the submerged area *ABK* will have the value $\gamma \cdot \frac{19}{90}\delta b^2$ as above shown; so that the bending moment at *B* would now be reduced to the lower value—

$$M' = \gamma(D - \tfrac{3}{5}\delta + \tfrac{19}{45}\delta)\frac{b^2}{2} = \frac{\gamma b^2}{2}(D - \tfrac{8}{45}\delta)$$

But it is obvious that when the supporting pressure is thus distributed, the deflection δ will have a value somewhat smaller than $\frac{wb^3}{8Ead^2}$. We may assume, however, that δ will be very nearly proportional to M and equal to $\frac{Mb^2}{2Ead^2}$, and that, whether δ be large or small, the area *ABK* will always be very nearly equal to $\frac{3}{5}\delta b$; so that finally we can write for the floating elastic pontoon (built originally straight)—

$$M_B = \frac{wb}{4} - \frac{2}{45}\frac{M_B \gamma b^4}{Ead^2} = \frac{\frac{1}{4}wb}{1 + \frac{2}{45} \cdot \frac{\gamma b^4}{Ead^2}} \quad . \quad . \qquad (5)$$

$$\delta = \frac{wb^3}{8} \cdot \frac{45}{45Ead^2 + 2\gamma b^4} \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

These formulæ are not quite mathematically correct, but the error involved is very much smaller than those which are almost certain to be contained in any hypothetical assumption that we we could make as to the elasticity of the pontoon.

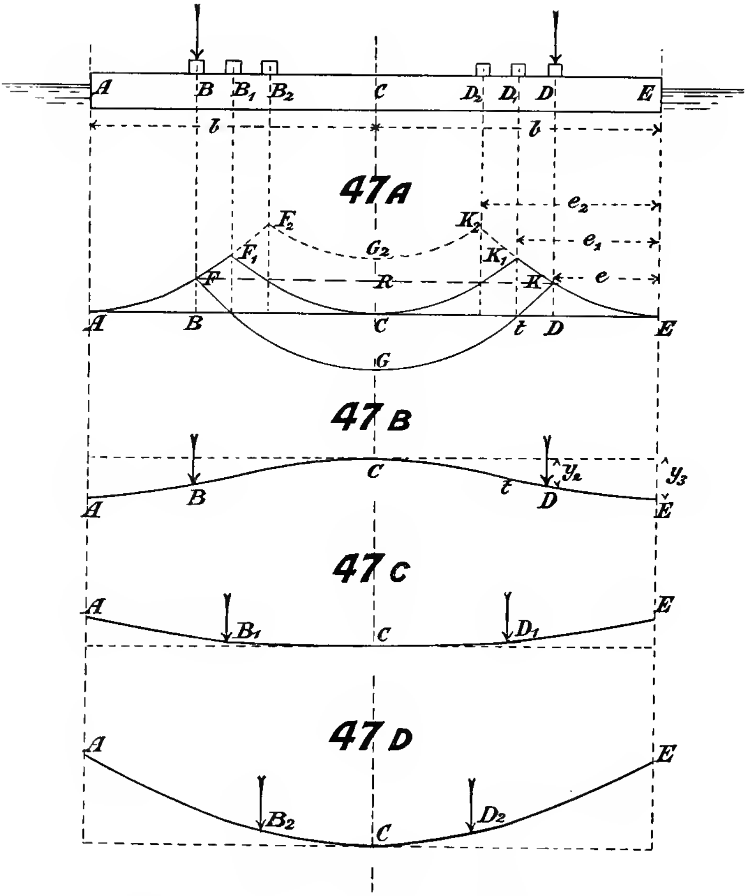
In many cases it would be sufficiently accurate to calculate M_B and δ by the simpler formulæ (2) and (4).

Art. 58. The Pontoon carrying a Pair of Equal Loads symmetrically disposed.—If the external load is carried upon two supports placed at equal distances from the centre, as sketched in Fig. 47, and if each support bears one-half of the total load W, the diagram of bending moments will have such a form as the curve *AFGKE*, or the curve $AF_2G_2K_2E$ in Fig. 47*a*, resembling the diagram of moments for a cantilever bridge under a uniform load; but the moment at the centre will largely depend upon the distance $\overline{BD}$ between the two supports, or the distance *e* of each support from the end of the pontoon.

Eliminating on one side the weight of the pontoon itself, which we suppose to be uniformly distributed, and on the other side a uniformly distributed buoyant pressure equal to that weight, we have only to deal with the external load W and a buoyant pressure of equal magnitude, whose intensity per lineal foot of the pontoon may, for the first approximation, be assumed to be nearly constant, neglecting the changes which may be due to elastic bending.

Then taking, as before, 1 foot of width, measured transversely to the line AE, let w be the total external load on that

FIG. 47



1 foot of width, so that for each foot of width the load at B and at D is $\frac{w}{2}$. It is obvious that the ends of the pontoon projecting beyond the supports B and D are simply inverted cantilevers under the uniform upward pressure, whose intensity per

foot lineal is $p = \frac{w}{2b}$, in which b is the half length of the pontoon.

From E to K, therefore, the curve of moments in Fig. 47*a* will have for its equation—

$$M = \frac{px^2}{2} = \frac{wx^2}{4b} \quad . \quad . \quad . \quad . \quad (7)$$

At the support D, where $x = e$, we shall have—

$$M_D = \frac{pe^2}{2} = \frac{we^2}{4b} \quad . \quad (8)$$

Up to this point, therefore, however great e may be, the curve will be just a portion of the curve already traced in Fig. 46*a*.

As soon as we get beyond the support, so that x is greater than e, we have for the rest of the curve, from K to the centre, the general equation—

$$M = \frac{px^2}{2} - \frac{w}{2}(x - e) = \frac{w}{2}\left(\frac{x^2}{2b} - x + e\right) \quad (9)$$

At the centre of the pontoon, where $x = b$, we have—

$$M_B = \frac{w}{2}\left(e - \frac{b}{2}\right) \quad . \quad . \quad . \quad . \quad (10)$$

The bending moment at the centre may, therefore, be either positive or negative, according to the position chosen for the supports B and D.

When the distance e, of either support from the end of the pontoon, is greater than one-half of b, or one-fourth of the whole length, the moment is positive at the centre, and positive throughout the whole length, the diagram being the curve $AF_2G_2K_2E$ in Fig. 47*a*, when $e_2 = \frac{5}{8}b$. In this case the pontoon is, of course, subjected to a sagging curvature throughout its whole length, as drawn in Fig. 47*d* to an exaggerated vertical scale.

If we make e equal to one-half of b, the bending moment at the centre will be zero, each half of the diagram consisting of a pair of parabolic curves, CK_1 and K_1E, and the stresses are then the same as in a pair of equal armed cantilevers. The curve of deflection is now almost straight at the centre, and also at the extreme ends, but more sharply curved at B_1 and at D_1, as shown in Fig. 47*c*. Each half of the curve is, in fact, similar to the curve already drawn in Fig. 46*b*. On the other hand, if we make e less than one-half of b, the bending moment at the centre will be negative, and there will be two points of contrary flexure somewhere between *B* and *D*. In the diagrams e is taken at $\frac{3}{8}b$, and the point of contrary flexure is then situated at t, half-way between *E* and *C*. The pontoon will now be sharply hogged in the middle, as shown in Fig. 47*b*. From t to *E* it will be slightly sagged, but notwithstanding this curvature, the inclination will still be a downward inclination, even at the extreme end; for the upward or downward inclination at *E* will be proportional to the algebraical sum of the negative and positive areas of the moment diagram, and when the support is at *D*, it is evident, by mere inspection, that the negative area *CtG* is greater than the positive area *EKt*.

The curves of deflection shown in Figs. 47*b*, 47*c*, and 47*d*, relate to the three cases in which e is taken to be $\frac{3}{8}$, $\frac{1}{2}$, and $\frac{5}{8}$ of the half-length b, and are all drawn to the same (exaggerated) vertical scale. To trace the curves in detail, the simplest plan would be to set off calculated ordinates y_1, y_2, etc., above or below a horizontal tangent drawn through the central point *C*; and at any section the ordinate y will be proportional to the algebraical sum of the moments of the positive and negative areas in the moment diagram, intercepted between that section and the centre line.

Thus, to find the deflection at *D* in Fig. 47*b*, let μ_1 denote the area *CtG* in Fig. 47*a*, and let ξ_1 be the horizontal distance of its centre of gravity from the point *D*; also let μ_2 and ξ_2 denote the same quantities for the positive area *DKt*; then the downward deflection at *D*, or the ordinate below the tangent, will be $-y_2 = \dfrac{\mu_1\xi_1 - \mu_2\xi_2}{\mathrm{EI}}$.

To obtain a general expression for the ordinates y_2 and y_3, the calculation may be simplified still further; for if we draw the horizontal line *RK*, and subtract the moment of the rectangle *RKCD* from the moment of the figure *RKG*, the difference will obviously be the same as the quantity $\mu_1\xi_1 - \mu_2\xi_2$ in the equation just above written.

A little consideration will suffice to show that we can treat the diagram in the same way in all the three cases, and thus we may obtain, by successive integration, a general expression for the deflection ordinate at each point of support, viz.—

$$y_2 = \frac{p(b-e)^2}{24\mathrm{EI}} \cdot \{6e^2 - 5(b-e)^2\} \quad . \quad . \quad . \quad (11)$$

and if we use z to denote the distance CD, or $b - e$, we have—

$$y_2 = \frac{pz^2}{24EI}(6e^2 - 5z^2) \quad . \quad . \quad . \quad . \quad (11a)$$

In the same way, we can summate the moments of the positive and negative areas reckoned about E, and we then have for the deflection ordinate at either end of the pontoon—

$$y_3 = \frac{p}{24EI}\{3e^4 + 6e^2z(2e + z) - z^3(8e + 5z)\} \quad . \quad (12)$$

Of course, the deflections calculated in this way are the deflections that would be produced under the bending stresses represented in Fig. 47*a*; and these bending stresses are themselves calculated upon the assumption that the buoyant pressure will be uniformly distributed. If we turn the deflection curve upside down, and then build the pontoon with an initial camber represented by the inverted curve, the pontoon will come to a straight line when floating under the given load; the buoyant pressure will then be uniformly distributed, and the bending stresses will have exactly the values above calculated.

On the other hand, if the pontoon is built initially straight, the imposition of the load upon the two supports will have the effect of producing a certain curvature, and in each half of the pontoon a certain change in the position of the centre of buoyancy. The extent of these changes will evidently depend, not only on the stiffness of the pontoon, but very largely also upon the disposition of the two supports; and the changes will be very small if we make the distance e equal to about seven-sixteenths of b.

Each individual case must, of course, be treated according to the actual dimensions and loads; and if it is found in any given case that the elastic deflection, as calculated by the formula, amounts to a considerable fraction of the mean draught, it may be desirable to introduce a correction for the altered distribution of the buoyant pressure. This may be done in the manner already illustrated in the last article, but for such a purpose it would probably be more convenient to work by graphic construction of the several curves rather than by analytical methods.

Art. 59. A Series of Concentrated Loads imposed at Three or More Points at Known Distances.—When the floating pontoon is loaded with a series of weights placed at equal or unequal distances, the diagram of moments may sometimes be similar in form to that which is so familiar to us in continuous girders; but the bending moments can be calculated without any of the difficulties which present themselves in the case of a continuous girder.

For if we know the actual distribution of the loads, we can find the distribution of the buoyant pressure within small limits of error; and when the vertical forces are all known, the bending moments can always be calculated by the general formula referred to in Art. 56.

Thus, for example, if the centre of gravity of the whole load coincides with the centre of the pontoon, we may assume that the buoyant pressure will be a constant quantity p per foot lineal, unless the elastic deflection is so great as to produce a sensible change in its distribution.

Then, to find the bending moment at any given section, whose distance from the right end of the pontoon is denoted by x, let W_1, W_2, etc., represent the weights of those loads which are imposed on the right-hand side of the section, and let X_1, X_2, etc., denote their horizontal distances from the given section; and we shall have for the bending moment—

$$\begin{aligned} M &= \frac{px^2}{2} - (W_1X_1 + W_2X_2 + \text{etc.}) \\ &= \frac{px^2}{2} - \Sigma(WX) \quad . \quad . \quad . \quad . \quad . \qquad (13) \end{aligned}$$

and for the vertical shearing force—

$$\begin{aligned} S &= px - (W_1 + W_2 + \text{etc.}) \\ &= px - \Sigma \,.\, W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14) \end{aligned}$$

It is hardly necessary here to apply the formulæ to any particular case; but it may be noted that the curve of moments will not always intersect the datum line between the loaded points, as it generally does between the piers of a continuous girder bridge.

Art. 60. Load uniformly distributed over a Portion of the Pontoon's Length.—It is sufficiently evident that a flat rectangular pontoon, carrying a load of *any* given intensity per foot lineal, which is uniformly distributed along its whole length, will float with a level bottom, and will undergo no bending stress whatever in the longitudinal plane.

But we may sometimes have to consider the stresses that will take effect when the load does not extend to the ends of the pontoon.

We will suppose that the load extends over a length $\overline{BD} = 2z$ in the middle of a pontoon, whose total length $AE = 2b$, as sketched in Fig. 48.

Eliminating again the weight of the pontoon, and also that portion of the buoyant pressure which goes to support it, we may consider only the external load, and the uniformly distributed buoyant pressure by which the external load is supported. If q denotes the intensity of the external load per lineal foot, we shall have for the distributed buoyant pressure the uniform value, or, at all events, the mean value $p = q\frac{z}{b} = q\frac{z}{e+z}$, in which e denotes the length of pontoon that has no load upon it, or the length DE in Fig. 48 at each end of the pontoon.

Then, to find the bending moment and the vertical shearing force at any section, whose distance from the right end of the pontoon is denoted by x, we may proceed as follows:—

For any distance x that is less than e, we have—

$$\mathrm{M} = \frac{px^2}{2};\ \text{and } \mathrm{S} = px$$

At the point D, where $x = e$, the bending moment will have the value $\mathrm{M}_\mathrm{D} = \frac{pe^2}{2}$; and this moment is represented by the ordinate DK in the diagram of Fig. 48a, the curve being a simple parabola from E to K. At the point D we shall have the greatest shearing force, its value being $\mathrm{S}_\mathrm{D} = pe$. To complete the curve of moments beyond K, we have, when x is greater than e—

$$\mathrm{M} = \frac{px^2}{2} - \frac{q(x-e)^2}{2};\ \text{and } \mathrm{S} = px - q(x-e)$$

At the centre of the pontoon we find the greatest bending moment, whose value will be—

$$\mathrm{M}_\mathrm{C} = \frac{pb^2}{2} - \frac{qz^2}{2} = \frac{peb}{2} = \frac{qez}{2} \qquad . \quad (15)$$

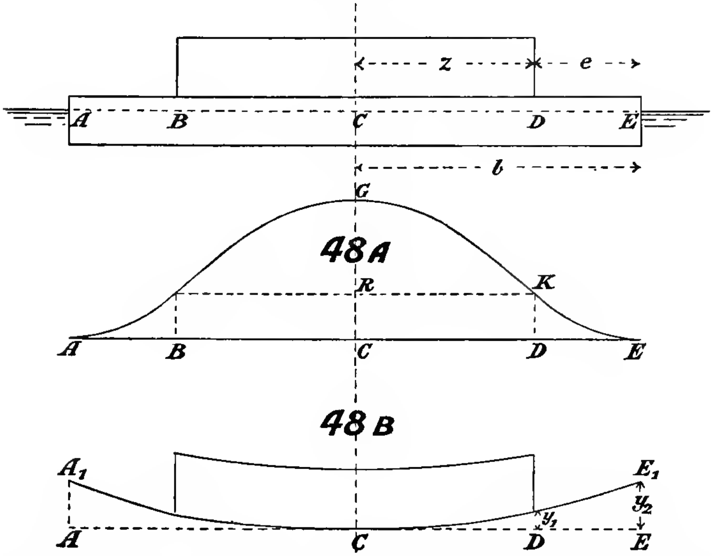
The whole diagram of moments will consist of a pair of reverse parabolic curves, something like an admiral's cocked hat.

The moments will everywhere be positive, and the pontoon will be sagged throughout its whole length, the curvature being sharpest in the middle.

To trace the curvature that would theoretically be produced by these bending stresses, we may draw a horizontal tangent through the central point *C*, as in Fig. 48*b*, and then we can calculate the ordinates at *D* and at *E* as follows:—

In the diagram of moments, the figure *GRK* will have the area $\frac{1}{3}pez^2$, while its moment about *K* will be $\frac{5}{24}pez^3$. The rectangle *CRKD* will have the area $\frac{1}{2}pe^2z$, and its moment about *K* will be $\frac{1}{4}pe^2z^2$. Adding these moments together and dividing by EI, we have for the upward ordinate at *D*, or the sagging deflection of the loaded portion—

FIG. 48



$$y_1 = \frac{pez^2}{24\text{EI}}(6e + 5z) \quad . \qquad . \quad (16)$$

Then reckoning the moments of the same areas about the point *E*, we have for the first $\frac{1}{24}pez^2(8e + 5z)$, and for the second $\frac{1}{24}pe^2z(12e + 6z)$, while the moment of the area *DKE* will be $\frac{1}{8}pe^4$. Therefore, the upward ordinate at *E*, or the sagging deflection on the whole length of the pontoon, will be—

$$y_3 = \frac{pe}{24\text{EI}}\{3e^3 + 6ez(2e + z) + z^2(8e + 5z)\} \quad . \quad (17)$$

The bending stresses and the calculated deflection will depend very much upon the length e, which is left uncovered by the load at each end. In fact, the bending moment at the centre is, as we have seen, $M_C = \frac{qze}{2}$, or equal to half the external load qz, multiplied by the length e. If the pontoon were fitted with water-tight bulkheads at D and B, and if the outer compartments were partly filled with water, leaving them with only as much buoyancy as would support their own weight, the bending stresses would be annulled. But if the full buoyancy of a long pontoon must be employed to carry a load which extends over a short portion of its length, the bending stresses will be very considerable.

In this case, the distribution of the buoyant force will, as in previous examples, be affected in some degree by the elastic deformation of the pontoon; and if that deformation is large, it may be necessary to apply the correction already referred to; but the formulæ above given will be nearly correct, as they stand, unless the first calculation shows that the deflection is a large quantity.

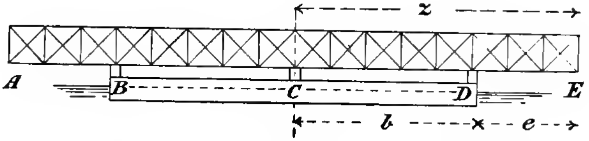
In this article, however, it has been assumed that the load is *uniformly distributed* over the given length $2z$, and that this distribution of the load will *not* be affected by the elastic deformation of the pontoon. The actual conditions might, in many cases, be quite in accordance with this assumption, but they will not be so in the cases next to be considered.

Art. 61. Pontoon carrying the Weight of a Long Girder.—When the external load consists in the weight of a girder, or of a ship which is capable of acting as a girder, we cannot, of course, assume that the load will be distributed along the pontoon just as it is distributed in the girder itself; but the forces transmitted from the girder to the pontoon, and the equal and opposite reactions by which the girder is supported, will depend upon the elastic deformations of both girder and pontoon. However, the nature of the problem will vary according to the conditions of the case, and these must first be defined.

In the present article we will assume that the half-length z of the girder is either equal to or greater than the half-length b of the pontoon, so that the girder overhangs at each end by a length $e = z - b$; also that the girder has a uniform weight q per foot lineal, and that its stiffness is such as to prevent any serious deformation under its own weight. In this case the deflection of the pontoon will evidently be limited by that of the girder, and will be so small that we may consider the draught to be uniform, and the uniformly distributed buoyant pressure that serves to support the external load will be $p = q \cdot \frac{b+e}{b}$. We can, therefore, calculate the moments due to the opposite forces with great exactness, and the method employed in the last article will suffice to determine those moments, though it may not suffice to determine separately the bending stresses in the girder and in the pontoon.

Thus, if x denotes the distance of any given point in the girder from the extreme end E, in Fig. 49, the curve of negative moments from E to D will be the parabolic segment in Fig. 49*a*, having for its ordinates the value $-\mathrm{M} = \frac{1}{2}qx^2$, while from D to C the negative moments will be expressed by $-\mathrm{M} = \frac{1}{2}qx^2 - \frac{1}{2}p(x - e)^2$. The form of the diagram will, therefore, be quite similar to that of Fig. 48*a*, but turned upside down, because the moments are now negative or hogging moments. These values of M must, in any case, represent the algebraical sum of the separate moments in the girder and in the pontoon, but the bending stress in each structure will depend upon conditions that have still to be considered.

FIG. 49



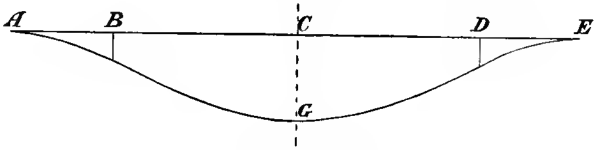


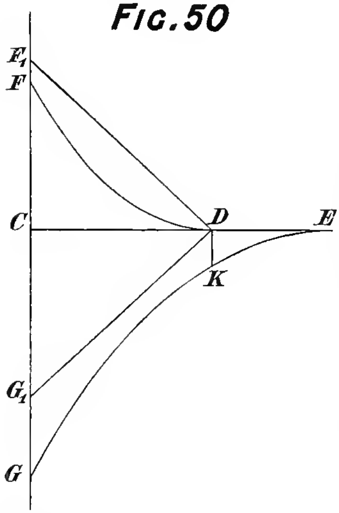
FIG. 49A

If the girder were not only supported by the pontoon, but also held down to it, at a series of points all along the back of the pontoon, we might perhaps regard the diagram of Fig. 49*a* as representing the moments that have to be shared between the girder and the pontoon in certain fixed proportions; and if the pontoon were excessively flexible, the diagram would represent very nearly the bending moments in the girder (supposing the girder to be nearly rigid). But these conditions will seldom occur in practice, and if the girder and the pontoon are not held together by any bonding connections, we must not assume that they will bend to exactly the same curvature so as to remain in contact along the whole length. The problem, however, will become quite definite when the points of contact are mechanically determined by the position of the supporting blocks.

To illustrate the question, we will here suppose that the supports are at the centre and at the two ends of the pontoon, as sketched in Fig. 49. Then the external load upon the pontoon will consist in three vertical forces, W_B, W_C, and W_D, whose sum must be equal to $2pb$, but whose individual magnitudes are unknown. But the same forces, reversed in direction, will constitute the reactions by which the girder is supported at *B*, *C*, and *D*; and a condition to be fulfilled is that the deflection ordinate y_1 at the point *D* must be the same for the girder as it is for the pontoon.

The moment diagrams for these several forces are drawn in Fig. 50, where the parabolic curve *EKG* represents the negative moments due to the weight of the girder, or $-M_1 = \frac{1}{2}qx^2$; while the curve *DF* represents the positive moments due to the buoyant pressure, or $M_2 = \frac{1}{2}p(x - e)^2$, and in each diagram the straight line DF_1 or DG_1 may represent the moments M_3 of the unknown vertical force W_D. The moment $CF_1 = CG_1$ has yet to be determined, but whatever value it may have, the deflection ordinate y_1 can be expressed, both for the girder and for the pontoon, by the aid of the following considerations:—

FIG. 50



Let μ_1 denote the area of the figure *DCGK* in Fig. 50, and ξ_1 the horizontal distance of its centre of gravity from *D*; also let μ_2 and ξ_2 denote the corresponding quantities for the triangle DCF_1 or for the equal triangle DCG_1; while μ_3 and ξ_3 shall denote the same quantities for the figure DCF (the areas being denoted in each case by μ, while ξ represents in each case the horizontal distance from D to the centre of gravity of the area). Then the deflection curve for the girder must have for its ordinate at D the value $y_0 = \frac{\mu_1\xi_1 - \mu_2\xi_2}{\mathrm{EI}_0}$, in which I_0 is the moment of inertia of the girder. In the same way, the curve of deflection for the pontoon must have for its ordinate at D the value $y = \frac{\mu_2\xi_2 - \mu_3\xi_3}{\mathrm{EI}}$, in which I is the moment of inertia of the pontoon.

It follows that if the girder and the pontoon bend together so as to preserve contact at C and at D, we must have $\frac{\mu_1\xi_1 - \mu_2\xi_2}{\mathrm{I}_0} = \frac{\mu_2\xi_2 - \mu_3\xi_3}{\mathrm{I}}$, and in consequence the triangle DCF_1 must have a particular height CF_1 and a particular area μ_2, which can be deduced without much difficulty.

Thus, if the moment of inertia of the girder is greater than that of the pontoon in the ratio $\mathrm{N} = \frac{\mathrm{I}_0}{\mathrm{I}}$, or if $\mathrm{I}_0 = \mathrm{NI}$, we shall have $\mu_1\xi_1 - \mu_2\xi_2 = \mathrm{N}(\mu_2\xi_2 - \mu_3\xi_3)$, whence we obtain quite easily—

$$\mu_2\xi_2 = \frac{\mu_1\xi_1 + \mathrm{N}(\mu_3\xi_3)}{\mathrm{N} + 1} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

To express these quantities in terms of the given loads and lengths, we can find, by integration or by geometric measurement, that the area $DCGK = \mu_1 = \frac{1}{6}q\{(b + e)^3 - e^3\}$, and its moment $= \mu_1\xi_1 = \frac{1}{24}q(6e^2b^2 + 8eb^3 + 3b^4) = \frac{1}{24}pb^3\left(3b + 5e + \frac{e^2}{b + e}\right)$.

For the area DCF, we have $\mu_3 = \frac{1}{6}pb^3$, and its moment will be $\mu_3\xi_3 = \frac{1}{8}pb^4$; and lastly, in the triangle DCF_1 or DCG_1, whose height CF_1 (or CG_1) represents the moment $\mathrm{W_D}b$ of the unknown force $\mathrm{W_D}$, we have for the area of the triangle the quantity $\mu_2 = \frac{1}{2}\mathrm{W_D}b^2$, and for its moment the expression $\mu_2\xi_2 = \frac{1}{3}\mathrm{W_D}b^3$.

Inserting these values in the equation (18) above, we have—

$$\frac{\mathrm{W_D}b^3}{3} = \frac{pb^3}{24(\mathrm{N} + 1)}\left(3\mathrm{N}b + 3b + 5e + \frac{e^2}{b + e}\right) \quad . \quad (19)$$

so that the height CG_1, representing the moment $W_D b$ of the unknown force at D, will have the value—

$$W_D b = \frac{pb}{8}\left\{3b + \frac{e}{N+1}\left(5 + \frac{e}{b+e}\right)\right\} \quad . \quad . \quad (20)$$

Then, setting off, in the diagram, this calculated height $CG_1 = CF_1$, and drawing the straight lines DF_1 and DG_1, the bending moments in the girder will be represented by the vertical ordinates intercepted between the line EDG_1 and the curve EKG; while the bending moments in the pontoon will, in like manner, be represented by the vertical ordinates intercepted between the straight line DF_1 and the curve DF.

It is obvious that the same principles would equally apply if it should happen that the line DF_1 intersects the curve, making the moment CF_1 less than the moment CF.

Thus it appears that the vertical force W_D, which acts in a downward direction upon the end of the pontoon, or the equal reaction which goes to support a certain part of the weight of the girder, may have varying values depending upon the ratio N as well as upon the overhang e; and so long as the girder touches the three supports at B, C, and D (in Fig. 49), its value will be determined by the equation—

$$W_D = \frac{p}{8}\left\{3b + \frac{e}{N+1}\left(5 + \frac{e}{b+e}\right)\right\} \quad . \quad . \quad (20a)$$

But it is obviously impossible that the reaction at D can ever be greater than one-half the weight of the girder, or the quantity $qz = pb$ (unless we conceive that the girder is tied down to the pontoon at C).

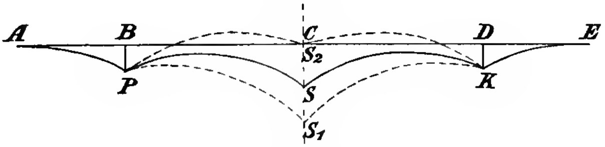
Subject to this limitation, which will presently be dealt with, the force W_D, as above found, may be used to calculate the bending moment at any point in the girder or in the pontoon, in the manner already referred to in Art. 56; or a diagram of moments may be constructed for each structure, by setting off, from a horizontal base-line, a series of ordinates equal to the heights intercepted between the straight lines and the curves in Fig. 50.

The two diagrams are drawn in Fig. 51 for a case in which the girder overhangs the pontoon by a small amount at each end, the overhang e being taken at $\frac{3}{8}z$, or $\frac{3}{4}b$. With this given amount of overhang, the full line $APSKE$ represents the curve of moments in the girder, and BTD the curve of moments in the pontoon, for the case when N = 1, or, in other words, when the stiffness of the girder is the same as that of the pontoon.

When the girder is stiffer than the pontoon, or N greater than unity, the negative bending moment at *C* becomes greater in the girder, while in the pontoon it becomes less, and perhaps changes its sign, the limit being reached when N is indefinitely great or the girder excessively stiff in comparison with the pontoon. In this extreme case, the diagrams approach to the limiting forms PS_1K and BT_1D, the latter being the curve of moments for a continuous girder of two equal spans.

On the other hand, if the pontoon were stiffer than the girder, the curves of bending moments would undergo a change in the opposite direction, and would approach to the forms shown by the dotted lines PS_2K and BT_2D, which are the

FIG. 51



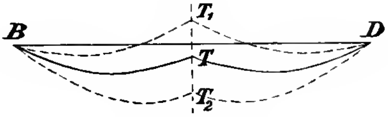


FIG. 51A

limiting curves for the extreme case when N is indefinitely small, or in other words, when the pontoon is excessively stiff and the girder excessively flexible. In this extreme case, the load is almost entirely carried at the ends of the pontoon, the force W_D being very little less than *pb*, as shown at once by inspection of formula (20*a*); and the bending stresses in the pontoon are almost the same as in an independent girder of the span *BD* under a uniform load *p*.

It is, perhaps, not very likely that a flat pontoon would, in practice, be stiffer than the girder or ship which it is intended to carry; but it may nevertheless be useful to pursue the question a little further for the sake of the converse problem which will have to be dealt with in the next article, and which is almost a parallel case to the one we are now considering.

Let us, therefore, suppose that the girder shown in Fig. 49 has now to be carried upon a pontoon whose length is *less* than five-eighths of the length of the girder, while its moment of inertia I may be either less or greater than that of the girder.

As we gradually reduce the half-length b of the pontoon, the quantity $pb = qz$ remains unaltered, but the bending stresses undergo various changes, and the force W_D is continuously increased, so that the load W_C which is carried at the central block becomes less and less. When the overhang e is so great that the force W_D, as calculated by formula (20a), reaches the value pb, the central load W_C becomes zero, and it is evidently impossible that W_D can have any greater value, however great may be the overhang. But if e is still further increased, the effect must be to raise the centre of the girder from contact with the block at *C*, leaving an arched space between the girder and the back of the pontoon. It will be useful to ascertain the conditions under which this could take place.

Evidently, if the girder should thus rise from its central seat, it would mean that the hogging deflection of the girder, on the length *BD*, is greater than the hogging deflection of the pontoon when the latter is carrying the load wholly upon its extreme ends. But, inasmuch as W_D is now equal to pb, or $q(b + e)$, the hogging deflection y_0 of the girder may now be written—

$$y_0 = \frac{\mu_1\xi_1 - \mu_2\xi_2}{\mathrm{EI}_0} = \frac{pb^3}{24z\mathrm{EI}_0}(6e^2 - 5b^2) = \frac{qb^2}{24\mathrm{EI}_0}(6e^2 - 5b^2)\ . \quad (21)$$

Hence it follows that, however rigid the pontoon may be, the girder cannot rise from the central block unless e is greater than $b\sqrt{\tfrac{5}{6}}$; or unless b is less than $0{\cdot}523z$.

When the length of the pontoon is only one-half, or less than one-half the length of the girder, the question will depend upon the relative values of the deflections y_0 and y, and these will depend upon the ratio N. By algebraical reduction it will easily be found that—

$$y_0 = y, \text{ when } \mathrm{N} = \frac{6e^2 - 5b^2}{5be + 5b^2} \quad . \quad . \quad . \quad (22)$$

Thus, for example, if we take a pontoon whose length is only three-eighths of the length of the girder, contact will be maintained at the three points, *B*, *C*, and *D*, unless the moment of inertia I_0 of the girder is less than that of the pontoon in the proportion of 7 to 8, or $N = \frac{I}{I_0} = \frac{7}{8}$.

In this article we have only examined the elementary case in which the load is carried upon three supports, and the question of contact at those three points. When the supports are more numerous, the problem becomes a great deal more complex, but will still admit of an approximate solution, either by an extension of the same method or by tentative adjustment of the moment diagrams.

Art. 62. Long Pontoon carrying a Short Ship or Girder.—In this article we will suppose that the ship or girder, carried by the pontoon, has a uniform weight q per foot lineal (leaving aside for the moment any question as to the actual distribution of weight in ships); and that the half-length z of the ship is either equal to or *less* than the half-length b of the pontoon, so that e represents the length of pontoon uncovered by the load at each end, as sketched in Fig. 52.

In such cases it is possible that the deflection of the pontoon might cause a sensible change in the distribution of the buoyant pressure; but for the present we will assume the deflection to be insignificant, and then we may say that the conditions are exactly the converse of those which we have been considering in the last article, the ship taking the place of the short pontoon, and the pontoon taking the place of the long girder, with all the forces reversed in direction but unaltered in magnitude.

Proceeding on these assumptions, the buoyant pressure concerned in carrying the external load will now be $p = \frac{qz}{b} = q\frac{b + e}{b}$; and if x denotes the distance of any point from the right end of the pontoon, the positive moment, at that point, due to the buoyant pressure, will be $M = \frac{1}{2}px^2$. This expression will define the curve *EK* in the diagram of moments due to the external forces p and q; and the remainder of the curve *KG* will have for its equation—

$$M = \tfrac{1}{2}px^2 - q(x - e)^2$$

For the case illustrated in Fig. 52, where $e = \frac{3}{5}z = \frac{3}{8}b$, the diagram of moments, as drawn in Fig. 52a, will, of course, be identical with that of Fig. 48a.

Of this diagram we may at once say that its ordinates, measured to the curve *AGKE*, must, at every point, represent the aglebraical sum of the two bending moments, which take effect in the ship and in the pontoon respectively; and this will be true generally, whatever may be the number or the position of the points of support. And further, if a very flexible ship were continuously supported upon a very rigid pontoon, the curve *AGKE* would give us, very nearly, the bending moments in the pontoon, the ship being in that case almost free from bending stress.

A floating dock of the "old Bermuda" type may be considerably stiffer than the ship which it carries; but in a flat pontoon without such deep side-walls, the moment of inertia I may be far less than the moment of inertia I_0 of the ship. In general, therefore, we cannot assume the pontoon to be rigid; and the actual distribution of the pressure between ship and pontoon will depend upon their elastic deflections, and therefore upon the ratio N.

Two questions may arise in practice—First, if we lay down the primary condition that the deflection and the bending stress in the ship are to be very small, or are to be nothing at all, then it is obvious that the pontoon will undergo the bending moments represented by the curve *AGKE*, and must consequently bend to a sagging curve, whose varying form has been illustrated in Fig. 48, and whose ordinates have been calculated in Art. 60; and the keel-blocks must be adjusted at *varying heights*, as given by the calculated ordinates. Without such an adjustment of the keel-blocks, or an equivalent initial camber of the pontoon, it would not be possible to attain the desired object; for as the pontoon bends the ship must follow to some extent, and would therefore be subjected to bending stress in some degree.

But secondly, if the keel-blocks are all laid to the same level upon the straight deck of an unstrained pontoon, and if the girder or the ship's keel is initially straight, we may have to inquire what deformations would take place under the load, and whether the ship would remain in contact with the support at *C*. Here the problem becomes very tedious if we assume a large number of supports; but if we take only three points of contact, the solution is given by the equations of Art. 61.

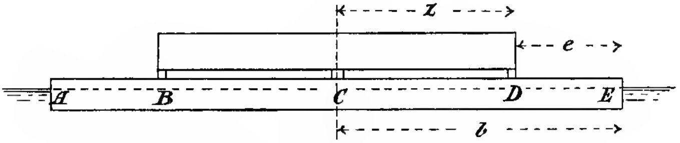
If, then, for the sake of simplicity, we consider the case of a short girder supported at *B*, *C*, and *D*, as sketched in Fig. 52, we can employ the equation (20*a*) to determine the unknown forces W_B, W_C, and W_D, whose values are not constant, but depend upon the relative length and upon the relative stiffness of the girder and of the pontoon. The expression for the unknown force at *D* or at *B* may now be written—

$$W_D = \frac{q}{8}\left\{3z + \frac{Ne}{N+1}\left(5 + \frac{e}{b}\right)\right\} \quad . \quad . \quad . \quad (23)$$

and the girder will only cease to have any bearing upon the central block when the conditions are such as to bring W_D up to the value $\frac{qz}{2}$.

Thus, for example, if we first take the case of a girder in which $z = \frac{5}{8}b$, we find that when $N = 1$, or $I_0 = I$, the force W_D amounts to 0·2883W (in which W is the total load); so that W_C is now equal to 0·5234W; and the curve of moments

FIG. 52



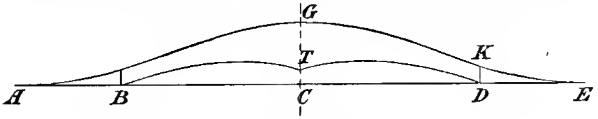


FIG. 52A

for the girder will now take the cusped form *BTD* in Fig. 52*a*, while the height intercepted between this curve and the curve *AGKE* gives us the bending moments in the pontoon. We also find that with this length of girder it is impossible for the pontoon to bend so as to withdraw the support at *C*; for if we take N at ever so great, or ever so small a value, the force W_D is always less than $\frac{1}{2}qz$.

If we next take the case of a shorter girder, in which $z = \frac{1}{2}b$, and assume first that $N = 1$, we shall find that the force W_D is now increased in magnitude, and amounts to 0·3596W, leaving only 0·2912W to be carried at the centre; and the central load will wholly disappear when $N = 10$, or when the girder is ten times as stiff as the pontoon.

Lastly, taking the case of a still shorter girder, in which *z* is only $\frac{3}{8}b$, and assuming first that $N = 1$, the force W_D is found to amount to $0{\cdot}4805W$, leaving only a very small fraction of the load to be carried at *C*; and if the girder is a little stiffer than the pontoon, W_C becomes equal to zero. The central support would, in fact, be withdrawn if N were greater than $\frac{8}{7}$, or I_0 greater than $\frac{8}{7}I$.

It may again be remarked that when the girder or the ship is continuously supported upon the pontoon, the bending stresses and the elastic deformations might be wholly eliminated if the pontoon were built with water-tight compartments at each end, so arranged that they could be partly filled when the pontoon is used for supporting a short ship.

Art. 63. Greatest Bending Moment.—Assuming that the load is always symmetrically placed upon the pontoon, extending right and left from the centre to some variable distance z less than the half-length b, the moments of the external forces will always be positive or "sagging" moments, and the greatest bending moment will occur at the centre *C*, as shown in Figs. 48*a* and 52*a*. The bending moment M_C at this point will depend on the distribution as well as the magnitude of the load.

For a given load W, the moment is $M = \frac{1}{2}We$, and is therefore zero when e is zero, or when the load is spread uniformly over the whole length; while its greatest value occurs when $e = b$, or when the whole load is concentrated at the centre *C*.

For a given load per foot lineal q, the moment is given by $M_C = \frac{1}{2}qez$. The bending moment is therefore zero when $z = 0$, and also when $z = b$; and it reaches its maximum value when $z = \frac{1}{2}b$, or when the load extends over one half the length of the pontoon, leaving one-fourth of the length uncovered at each end.

When the load consists in the actual weight of a ship, whose half-length (measured on the load water-line) is denoted by z, the weight will not generally be distributed uniformly over that length. The centre of gravity of the whole vessel may probably coincide pretty nearly with the midship section, and we may take $\frac{1}{2}W$ to represent very nearly the weight of the fore body or of the after body.

Then, if ζ denotes the horizontal distance from the midship section to the centre of gravity of the fore body (or of the after body), the moment at *C* will be given by—

$$M_C = \tfrac{1}{2}W\left(\frac{b}{2} - \zeta\right) \quad . \quad . \quad . \quad . \quad . \quad (24)$$

In general, it would be a tedious matter to ascertain the length ζ, although there would be no great difficulty in finding the horizontal distance ζ_1 from the midship section to the centre of buoyancy of the fore body, or of the after body.

If ζ were exactly equal to ζ_1, the ship would, of course, suffer no bending stress at *C* when floating in calm water; but when the vessel is built with fine ends, ζ is generally greater than ζ_1, and in calm water there will be a certain hogging moment at the centre, whose value will be $\frac{1}{2}W(\zeta - \zeta_1)$.

Rankine estimates that, in ordinary cases, this hogging moment may be taken roughly as varying between $\frac{1}{30}Wz$ and $\frac{1}{20}Wz$; whence it would follow that the distance ζ in formula (24) would vary between the values $\zeta_1 + \frac{1}{15}z$ and $\zeta_1 + \frac{1}{10}z$; and this estimate might suffice to determine the greatest moment M_c by means of formula (24), when ζ_1 has been found by measurement of the ship's lines, or estimated by reference to the average practice in shipbuilding.

Art. 64. Greatest Bending Moment in the Ship and in the Pontoon.—When the half-length z of the ship is less than the half-length b of the pontoon, the bending moment M_c in formula (24) will represent the bending stress that has to be shared in some proportions between the ship and the pontoon. And if the keel-blocks are spaced at short intervals, it may perhaps be assumed, without much error, that, *at and near the centre*, the pontoon will so deflect in company with the ship as to assume very nearly the same curvature. Then if I_1 is the moment of inertia of the pontoon, and I_2 that of the ship, the greatest bending moment in the pontoon will be approximately—

$$M_1 = M_C \cdot \frac{I_1}{I_1 + I_2} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the greatest bending moment in the ship—

$$M_2 = M_C \cdot \frac{I_2}{I_1 + I_2} \qquad (26)$$

APPENDIX A.

EXPERIMENTAL MEASUREMENT OF THE DISPLACING FORCES.

1. **Equilibrated Fluid Arch or Vertical Bend.**—These illustrative experiments were designed to form part of the laboratory work in University College, Dundee.

The tubular arch was formed carefully to a segmental curve with a radius of 13 feet, the span of the arch being 10 feet and the rise 1 foot. The tubes were either of drawn steel or of copper. The copper tube had an internal diameter of 1·00 inch, and was fitted at each end with a sliding expansion joint formed upon the end of a short brass cylinder, which was mounted upon trunnions, so that the cylinder could adjust itself to the line of the pipe.

To prevent the copper tube from carrying any portion of the load, as an arch, the sliding neck of the expansion joint was turned down to an external diameter of 1 inch, exactly equal to the internal diameter of the pipe; so that the pipe was entirely free from any longitudinal stress, so far as the hydraulic pressure was concerned. The slight stress that might be due to collar friction, in one direction or the other, was eliminated in the usual way by measuring its value in both directions.

As there could be no longitudinal stress in the arched tube, it follows that, however great the load might be, there would be none of that elastic deformation (due to compression) which has usually to be reckoned with in an arched rib.

At the crown, the arch was free to rise or fall between a pair of vertical guide-bars, which were placed as shown in Fig. 53; and stop-pins were inserted through the guide-bars to limit the deflection in each direction.

The load was applied by a series of radial stirrups placed at equal distances around the segment, a scale-pan being suspended from each; and the stirrups were kept in radial position by a bridle-wire. The scale-pans were so loaded as to bring the same radial pull upon each stirrup.

The water was admitted by a central joint in the trunnion of one cylinder, and was supplied under variable pressure from the hydraulic cylinder of a Wicksteed testing machine, through a flexible copper pipe of small diameter. The pressure was varied by running out the load upon the testing machine, and was measured by a large-scale pressure-gauge made for the purpose by Messrs. Schaeffer and Budenberg, and attached to the trunnion of the other cylinder.

To prevent the pipe from carrying any portion of the load as a girder, the figure of the unstrained arch was carefully measured, and the true position of the crown (or the rise of the unstrained arch) was marked as a zero point upon the vertical guides, so that deflections could be measured above and below this point; and the readings presently to be mentioned were taken when the observed deflection was zero. As the pipe was found to give a deflection of 1 inch for a central load of 23 lbs., while the actual deflection could easily be read to within one-fortieth of that quantity, it may be assumed that no appreciable part of the load was carried by the transverse stiffness, and that practically the whole load was borne by the water-arch, although the readings were of course interfered with by the collar friction.

The following readings were taken in September of 1896:—

	No. 1.	No. 2.	No. 3.
Radial load n_1 in pounds per lineal foot of the arc ...	21·25	42·5	63·75
Equivalent suspended load on the whole segment, pounds	229	458	687
Equivalent distributed load on a parabolic arch, pounds	220	440	660
Hydraulic pressure in pounds per square inch	350	700	1080
Observed deflection	0	0	0

In each of the three cases the load was equilibrated by the fluid pressure, but the different loads were intended to illustrate three different kinds of equilibrium, and the different behaviour of the elastic arch in each case, as presently to be described.

2. Experimental Buckling of the Straight Pipe.—The pipes used for the illustration of this question were either of drawn steel or of copper, and the greatest care was taken to ensure perfect straightness in the unstrained pipe. The brass cylinders that had been used for the ends of the arch were again employed as expansion joints, the pipe being laid between them in a horizontal position, as shown in Fig. 54. The cylinders being free to turn upon the trunnions, the pipe was free to deflect in either direction in the vertical plane, and was thus placed under conditions analogous to those of a column with hinged ends. The weight of the pipe, whether full or empty, was in all cases counterbalanced.

The length from trunnion to trunnion was 10 feet. The vertical deflection was limited by stop-pins inserted in the guide-bars as before. The elasticity of the empty pipe was first ascertained by applying, at the centre, the pull of a spring balance; and the same instrument was afterwards used to illustrate (differentially) the

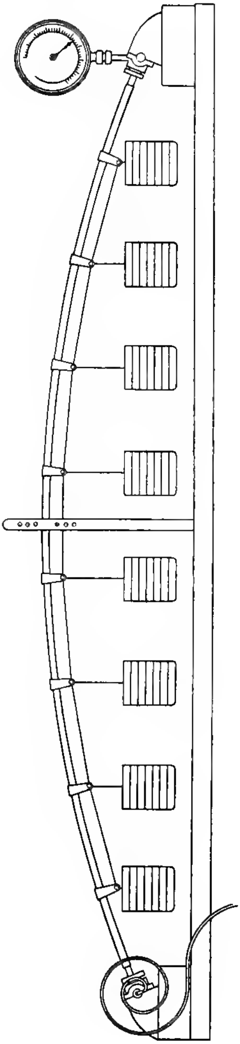


FIG. 53

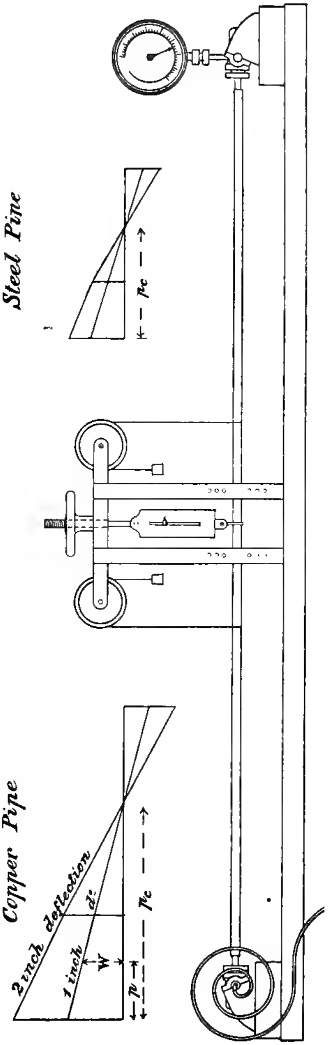


FIG. 54

FIG. 54 B Steel Pipe

FIG. 54 A Copper Pipe

effect of the displacing force at different curvatures and under different hydraulic pressures.

In the case of the copper pipe, the internal diameter was exactly equal to the bore of the stuffing-box gland, or 1·00 inch; so that the tubular column was subjected to no compressive stress whatever, and was buckled wholly by the lateral displacing force.

The external diameter being 1·17 inch, the moment of inertia, in inch units, amounts to 0·043, nearly; but the quantity EI was experimentally found by transverse bending under the central pull of the spring balance.

The elastic beam having a span of 120 inches, will exhibit, according to the usual formula, a deflection of $\frac{36000\text{W}}{\text{EI}}$, under the central load W; and as the spring-reading for a deflection of 1 inch was found to be 22 lbs., it follows that EI must amount to 792,000. From this experimental value of EI, it is easy to find the quantity $\text{EI}\frac{\pi^2}{l^2}$, which should theoretically determine the axial fluid stress that is just enough to buckle the column, or the equivalent hydraulic pressure $p = \frac{4\pi\text{EI}}{l^2 d^2}$; and the critical pressure for the copper pipe works out accordingly at 691 lbs. per square inch.

This calculation having been made beforehand, the pipe was charged to a hydraulic pressure of 690 lbs. per square inch, and was then found to exhibit very clearly that condition of indifferent equilibrium which characterizes the elastic column when loaded with its critical load.

The displacing force is now just sufficient to balance the elastic reaction of the pipe, no matter whether the deflection be small or great, within moderate limits. To produce a lateral deflection of 1 inch or of 2 inches, requires theoretically no exterior lateral force, and when the spring balance was applied for the purpose, the reading was too small to be observed. But to restore the pipe to straightness from a deflection of 1 inch or of 2 inches, requires also no exterior force; and therefore no force is required to *prevent* the pipe from moving laterally. The pipe, in fact, would have no tendency to move from any position, straight or curved, in which it may be placed; and therefore the frictional resistance of the trunnions is amply sufficient to prevent any motion in one direction or another. It is obvious, however, that when the critical hydraulic pressure is exceeded, the pipe will be in unstable equilibrium, and must be buckled, unless that is prevented by the application of an exterior force in the *contrary* direction.

To obtain a more exact measurement of the critical point, the hydraulic pressure was first reduced to 345 lbs., and then raised to three times that amount; and the positive and negative forces which were required to equilibrate the pipe under these pressures were measured by the spring balance, which was applied in such a direction as to aid the displacing force when that was insufficient, or to counteract it when it was more than sufficient to balance the elastic reaction of the pipe. The readings were taken successively for measured deflections of 1 inch and of 2 inches; and for each deflection the observed pressure p and spring-reading W are plotted as co-ordinates in the diagram shown in Fig. 54*a*. The diagram is a nearly straight line intersecting the base-line at a certain point, which determines the critical pressure p_c a little more exactly than it was possible to fix it by direct observation. And whether the line is plotted by the co-ordinated readings taken for a deflection of 1 inch, or for a deflection of 2 inches, it crosses the datum line at the same point, which is the point of indifferent equilibrium. The ordinates of the diagram on the negative side, as taken for successive deflections at any one pressure, serve to illustrate the fact that, when the hydraulic pressure is at all greater than p_c, the external force that must be applied in order to prevent the deflection from increasing, is itself proportional to the instantaneous deflection, and as the pipe bends, this force must be continuously augmented.

The following readings have been recorded:—

COPPER PIPE (September, 1896).

External diameter, 1·170 inch; internal diameter, 1·00 inch.

Central Deflection, 1 *inch*.

	lbs.	lbs.	lbs.	lbs.
Hydraulic pressure	0	345	690	1035
Spring-reading	+ 22·3	+ 11·5	0	− 10

Central Deflection, 2 *inches*.

	lbs.	lbs.	lbs.	lbs.
Hydraulic pressure	0	345	690	1035
Spring-reading	+ 43·8	+ 24	0	− 20

STEEL PIPE (January 5, 1897).

External diameter, 1·00 inch; internal diameter, 0·90 inch.

Central Deflection, 1 *inch*.

	lbs.	lbs.	lbs.	lbs.
Hydraulic pressure	0	180	360	540
Spring-reading	+ 11·25	+ 6	+ 1	− 8

Central Deflection, 2 *inches*.

	lbs.	lbs.	lbs.	lbs.
Hydraulic pressure	0	180	360	540
Spring-reading	+ 22·1	+ 13·1	− 1·5	− 13·7

More accurate results may be obtained if knife-edges are used for the bearings of the cylinders. The tests are sufficient to corroborate, if that were needed, the theorems contained in Chapter IV., but they are not really necessary for the purpose of *demonstrating* the truth of those theorems. To the mind of the student, however, they convey a clear impression as to the nature of the mechanical actions and reactions that are concerned.

3. Buckling of the Fluid Arch.—When the tubular arch is exactly equilibrated under the action of the exterior normal load n_e, and the equal and opposite displacing force n, uniformly distributed around the segment, the elasticity of the pipe resists any deflection at the crown, upwards or downwards.

And whether the elastic reaction that would be evoked by any given reflection be measured by a central force W (such as that applied by the spring balance), or whether it be expressed in terms of a uniformly distributed force n_1, that reaction will be nearly proportional to the vertical deflection δ.

For any given deflection, or given increment Δh in the rise of the arch, measured from its chord-line, we can find the corresponding increment Δn in the displacing force, due to the increased curvature.

If the elastic reaction n_1 is greater than the increment Δn, the arch will be in stable equilibrium, and the crown will come to the zero mark upon the guide-bars, returning always to that position if, for a moment, it is temporarily displaced.

But if the increment Δn, positive or negative, is greater than n_1, while n_e remains constant, the arch will be in unstable equilibrium, and will show a tendency to move away from the zero mark, the tendency becoming stronger the further it moves; and whether it happens to be started in an upward or a downward direction, its motion will only be arrested by the stop-pin. This is, in fact, a buckling of the arch in the vertical plane upon a free length equal to the entire length of the segment.

The increment Δn is itself proportional to the hydraulic pressure p, and therefore the equilibrium becomes unstable as soon as p exceeds a certain limiting value p_c.

The three conditions of equilibrium are illustrated by the three tests already referred to. With a load of $6\frac{1}{4}$ lbs. per foot, the arch is equilibrated under a pressure of about 1080 lbs. per square inch, but the equilibrium is unstable in the sense that has been described.

When the load is 42 lbs. per foot, the arch is equilibrated under a pressure of about 700 lbs., the equilibrium being sensibly indifferent; but when the pressure was raised to 750 lbs., the arch with its suspended load was continuously lifted, until it withdrew itself from the tubular abutments altogether.

It should perhaps be remarked here that the loads which are given in the table on p. 136 as the "equivalent" loads on a segmental or a parabolic arch, are the distributed loads that would produce the same *horizontal thrust* at the crown as that due to the radial forces which were actually applied. There is, of course, no distributed load that can be equivalent *in all respects* a to distributed radial pressure.

APPENDIX B.

FLUID CATENARY.

It has already been shown, in Art. 26, that when the effective building material of the arch consists of a fluid under uniform pressure, the entire load being carried by the fluid arch, and no part of it by the surrounding tubular arch of metal, the internal sectional area of the tube must at all points be proportional to sec θ. Hence the weight of the fluid per unit of arc-length will also be proportional to sec θ, while its weight per unit of horizontal length (x) will be proportional to $\sec^2 \theta$, or to $1 + \tan^2 \theta$.

The same considerations imply that the diameter d of the tube will at all points be proportional to $\sqrt{\sec \theta}$; and if the tube is so built that the working stress in hoop-tension shall have any determined value f per square inch of metal section, the thickness of the tube will at all points be proportional to the diameter or to $\sqrt{\sec \theta}$. If, then, we assume that the thickness of the tube will be small in comparison with its diameter, its weight per unit of arc-length will be very nearly proportional to πd multiplied by the thickness, and therefore proportional to sec θ; so that this part of the load, like the weight of the fluid itself, will have an intensity per unit of horizontal length, proportional to $\sec^2 \theta$.

Therefore, in the case which we are supposing, wherein the arch is equilibrated under its own weight, including the fluid and its containing envelope, the entire load per unit of x is proportional to $\sec^2 \theta$, and the form of the arch can be determined accordingly.

If the tube were formed in a series of short, straight lengths, each length being cylindrical in form, and if AB and BC in Fig. 55*b* are taken to represent the centre lines of two such contiguous lengths, having the inclinations θ_1 and θ_2, the vertical force BG, in the parallelogram $ABCG$, may be taken as consisting of one-half of the weight of the tube AB and one-half of BC; while in the same force diagram, AB_1 or BC_1 will represent the horizontal thrust at the crown, and AB the inclined thrust in the first length.

If each side of the polygon subtends the same horizontal length

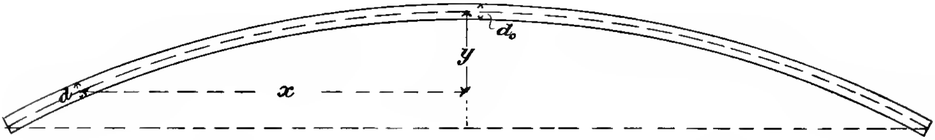
$\Delta x = AB_1 = BC_1$, it will follow that the weight of each pipe must be proportional to $\Delta x \sec^2 \theta$, and at each joint of the polygon the vertical BG will be proportional to the mean of the squares of AB and BC.

The curve will be the same as that of a suspension chain, when the load consists only in the weight of the chain, and when the links are made of uniform strength—*i.e.* not of uniform section, but with a section proportional to the stress.

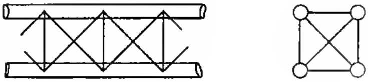
For in the arch constructed as we have supposed, the weight w per foot of arc-length is always proportional to the direct thrust S, so that $S = mw$, m being a constant multiplier.

When the tube is charged with hydraulic pressure, the weight of the water will probably form a large part of the total weight w, and the multiplier m will vary in each case according to the diameter d, the pressure p, and the working strength f. If, however, the fluid is compressed air, or any elastic gas, the fluid weight will be very small in itself, and will form a constant fraction of the weight of the pipe, so that m will be theoretically independent of p or d (within certain limits), and will depend only on the working strength f.

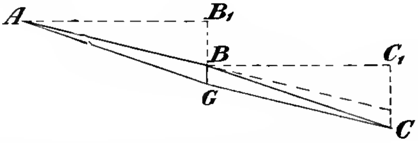
FIG. 55.



55A



55B



The curve will then have one definite figure expressed by the function—

$$y = m \text{ hyp. log.} \left(\sec \frac{x}{m} \right)$$

in which x and y are the co-ordinates shown in Fig. 55; and m will be the radius of curvature at the crown.

Thus, if we assume for a steel tube a uniform working stress of 6 tons per square inch, its thickness in inches will be $\frac{pd}{12 \times 2240}$; and as the weight of each square inch of steel section is about 3·36 lbs. per foot lineal, the weight of the tube per lineal foot would be $\frac{3\cdot36\pi d^2 p}{12 \times 2240}$, which is equal to $\frac{1}{2000}$th of the actual stress S, or $\frac{pd^2\pi}{4}$. For a tube of drawn steel, therefore, the curve of the equilibrated arch would have a radius of about 2000 feet at the crown.

We should, however, add to this weight of tube about 7 per cent. for the weight of the contained air under any given pressure p; and if, for the sake of example, we further add 53 per cent. for the estimated supplementary weight of connections, etc. (including loss of strength at rivet-holes), the multiplier m will be reduced to 1250.

With this estimated weight of the built tube, we should have the following values for the co-ordinates x and y in Fig. 55. For $m = 1250$.

x.	Span $= 2x$.	Rise $= y$.	Ratio $\left(\frac{d}{d_0}\right)^2$
100	200′	4·00	1·00
200	400′	16·08	1·01
300	600′	36·35	1·03
400	800′	65·11	1·05
500	1000′	102·76	1·08
600	1200′	149·87	1·12
700	1400′	204·63	1·18

Thus a tubular arch of 1200 feet span, with a rise of one-eighth of the span, would suffer no compressive stress from its own weight, so long as the tube was charged to a pneumatic pressure p sufficient to produce a hoop-tension of 6 tons per square inch.

So far as the weight of the *tubes* is concerned, the multiplier m would be nearly the same for different diameters and different thicknesses of pipe (within moderate limits); and the calculated co-ordinates would have the same value, whether the arch consisted of one large tube or of several smaller tubes, grouped together as in Fig. 55*a*, except that the weight of bracing would then have to be taken into account.

APPENDIX C.

EXPERIMENTS ON THE TENSILE STRENGTH OF SOCKET JOINTS.

The object of these tests was to find, by actual experiment, the resistance which a socket joint offers to a direct longitudinal pull inflicted by the pipe itself, just as it would be at the joints of a bend, for example, under the action of the displacing force.

The pull was, therefore, applied to the pipe axially, and was measured by means of a Wicksteed testing machine, as the measurement obtained in this way was more accurate than any that could be made by means of hydraulic pressure.

It is possible that the resistance of the lead joint would be affected, to some slight extent, by the presence or absence of internal hydraulic pressure—either diminished by water lubrication or increased by lateral pressure upon the lead—while the egress of the lead would, no doubt, be aided by the fluid pressure upon the annular end; but it does not seem probable that any of these forces or resistances would very much help or hinder the action of the great axial stress that is sometimes transmitted through the pipe itself. At any rate, the tests were made *without* the presence of any internal fluid pressure, and for the purpose that has been mentioned.

The socket joints that were thus tested included several different forms, such as are commonly employed in water-pipes of large diameter; but as the power of the testing machine was limited to 50 tons, the sockets were necessarily limited to the small diameter of 3·5 inches in the bore, although in other respects the sections were such as would be used in the largest pipes; and it seems reasonable to suppose that, with any given section, the resistance would be nearly proportional to the circumference measured at the line of shear.

The pipes and sockets were of cast-iron without any bituminous or other coating, and in general the surfaces were left as they came from the mould; but in one case the socket was smoothly bored, as specified in the description of joint No. 2*c*.

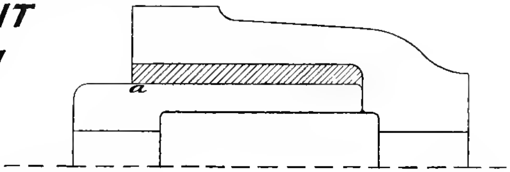
In all cases the lead was well driven up by a caulking tool at the mouth of the socket. Whenever gaskin was employed, a considerable movement of the pipe took place, during its compression, without starting the lead; but in all cases the load here recorded is the load under which the lead began to move, and from this point the resistance generally decreased as the joint slowly drew out.

To ascertain the ultimate resistance, it was necessary to apply the load with extreme slowness.

SOCKET JOINT No. 1.

Length of socket, 4·0 inches, entirely filled with lead. Diameter of socket, 3·5 inches, plain cylindrical form. Diameter of pipe, 2·92 inches, plain cylindrical form.

JOINT No 1



Greatest resistance	... ...	5·7 tons.
,, ,,	per inch of circumference ...	12·4 cwt.
,, ,,	per square inch of area at *a* ...	3·1 ,,

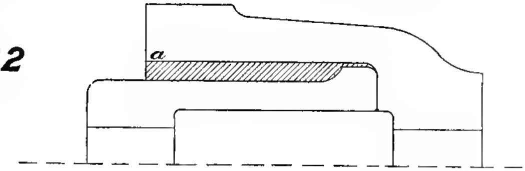
SOCKET JOINT No. 2.

The same socket as in No. 1—no gaskin used. Pipe 2·92 inches outside diameter, with a shoulder 3·25 inches in diameter, and $\frac{5}{8}$ inch thick.

Depth of lead in joint No. 2*a*, $3\frac{3}{8}$ inches.
,, ,, ,, No. 2*b*, $1\frac{1}{2}$,,
,, ,, ,, No. 2*c*, $1\frac{1}{2}$,,

The socket of No. 2*c* was bored to a smooth surface.

No 2

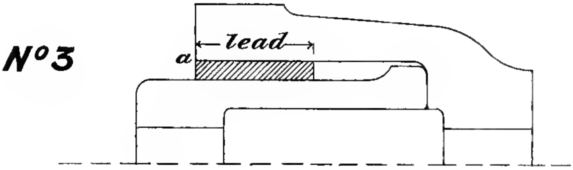


	2*a*.	2*b*.	2*c*.
Greatest resistance in tons	30 5	11·4	17·2
,, ,, cwt. per inch of circumference	55·5	20·8	31·4
,, ,, cwt. per square inch of area at *a*	16·5	13·9	20·9

SOCKET JOINT No. 3.

The same shouldered pipe and cylindrical socket, shown in joint No. 2, were used again in this case; but the joint was made with a packing of gaskin, varying from 1 inch to 1⅝ inch in depth, the remainder of the socket being filled with lead.

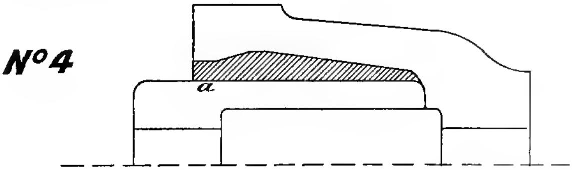
Depth of lead in joint No. 3*a*, 1¾ inches.
" " " No. 3*b*, 2⅛ "
" " " No. 3*c*, 2⅜ "



	2*a*.	2*b*.	2*c*.
Greatest resistance in tons	6·82	5·00	6·25
" " cwt. per inch of circumference at *a*	12·4	9·1	11·4
" " cwt. per square inch of area at *a*	7·1	3·8	4·8

SOCKET JOINT No. 4.

Depth of socket, 4·0 inches, entirely filled with lead. Socket formed with a recess 3·9 inches internal diameter. Plain cylindrical pipe without shoulder.

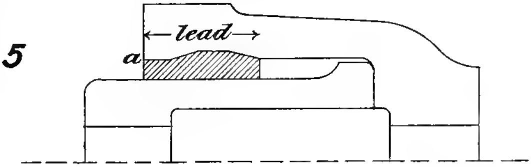


Greatest resistance in tons	5·5 tons.
" " cwt. per inch of circumference at *a*	12·0 cwt.
" " cwt. per square inch of area at *a*	3·0 "

SOCKET JOINT No. 5.

Socket formed with a recess 3·9 inches internal diameter. Pipe cylindrical, with an external diameter of 2·9 inches, and with a shoulder. Joint made with gaskin $1\frac{3}{8}$ inch in depth, and lead 2 inches deep.

No 5



Greatest resistance	... ...	17 tons.
" " per inch of circumference at *a*	...	30·9 cwt.
" " per square inch of area at *a*	...	15·4 "

Remarks.—In all tests that were made with a plain cylindrical pipe (without any shoulder), the lead was left wholly in the socket, undisturbed by the withdrawal of the pipe. The resistance of joint No. 4 was, therefore, not at all increased by the recess in the socket. In this joint, as in joint No. 1, the resistance consisted only in the adhesion between the lead and the cylindrical surface of the pipe; and in both cases alike it amounts to about 3 cwt. per square inch of that cylindrical surface.

In all tests that were made with joint No. 2, the lead was sheared through its whole length by the shoulder at the end of the pipe, leaving a thin lining of lead which covered the whole interior surface of the socket, and which was only withdrawn (along with the rest of the lead) when the shoulder emerged from the socket. The lead appeared to have been wedged outwards by the conical neck of the shoulder, producing an adhesion with the interior surface of the socket sufficient to prevent the lead from moving forward; and this adhesion was apparently not diminished in the case of No. 2*c*, when the socket was bored to a smooth cylindrical surface. In each case the resistance per square inch of the sheared area was very much greater than in joints Nos. 1 and 4.

In all the tests made with joint No. 3, the gaskin was first compressed into a hard wafer, and then the lead was forced bodily forwards. The insertion of the compressible gaskin between the lead and the shoulder prevented the wedging of the lead radially outwards, and had the effect of greatly reducing the tensile strength of the joint. The resistance was affected in some degree by the oil which exuded from the gaskin under pressure, and in some instances was very little greater than the resistance shown by Nos. 1 and 4. The lead was not sheared nor in any way distorted.

In the case of joint No. 5, it was, of course, impossible that the insertion of the gaskin could be attended with these consequences; for the lead that filled the recess in the socket could not be forced forwards except by a plastic flow of the material or a shearing of the lead by the advancing wafer of hardened gaskin.

It would appear, therefore, that when the joints are to be made with any kind of elastic packing, the recessed socket offers the means of obtaining a high degree of longitudinal strength, when it is employed in combination with the shouldered pipe. The same, or nearly the same strength, was shown in the plain cylindrical socket, so long as the joint was made wholly with lead; but the remarkable adhesion that was shown in joint No. 2, disappeared altogether when the gaskin was introduced.

APPENDIX D.

CURVE OF DEFLECTION IN A PIPE BUCKLING UNDER INTERNAL PRESSURE.

(*Referred to in Art.* 22.)

REGARDING the pipe as a beam of uniform section supported at each end of the span l, let us assume hypothetically that it is subjected to a distributed load or transverse pressure of varying intensity, proportional at all points to the ordinates in the curve of sines, as sketched in diagram A of Fig. 24. That is to say, if k denotes the radius of the circle whose semi-circumference is l, or $k = \frac{l}{\pi}$, and if n_{B} denotes the maximum load per foot at the centre of the span, assume that the load per foot at any other point is $n = n_{\mathrm{B}} \sin \frac{x}{k}$.

Each half of the diagram A will then have the area $n_{\mathrm{B}}k = n_{\mathrm{B}} \frac{l}{\pi}$, and will represent one-half of the total distributed load, or the load transferred to each abutment.

The centre of gravity of the same area, or the centre of gravity of the half-load will be situated at a point whose horizontal distance from the abutment is exactly k or $\frac{l}{\pi}$.

At the centre of the span, the moment of the abutment's reaction will therefore be $n_{\mathrm{B}}k\frac{l}{2}$; while the contrary moment of the distributed pressure on the half span will be $n_{\mathrm{B}}k\left(\frac{l}{2} - k\right)$; and we shall have for the bending moment at the centre of the span—

$$\mathrm{M}_{\mathrm{B}} = n_{\mathrm{B}}k^2 = n_{\mathrm{B}}\frac{l^2}{\pi^2} \quad . \quad . \quad . \quad . \quad (1)$$

If the curve of bending moments is traced out in detail, whether by analytical or graphic methods, it will be found that the bending moment M at any other point in the span is expressed by—

$$M = M_B \sin \frac{x}{k} = nk^2 \qquad . \quad . \quad . \qquad (2)$$

so that the curve of moments sketched in diagram *C* of Fig. 24 will be a curve of sines, and similar to diagram *A* drawn to some other vertical scale.

In an elastic beam of uniform section, the curve of deflection may always be constructed from the diagram of moments in the same way that the curve of moments is constructed from a diagram of the load. The buckling deflection of this cylindrical tube may, of course, take place in *any* direction, upwards, downwards, or laterally towards either side; and to get rid of the idea of *weight*, which has nothing to do with the question, we may regard the curve of deflection sketched in diagram *E* of Fig. 24 as a ground plan. But in whatever plane the deflection may take place, it is obvious that the transverse displacing force due to the curvature of the tube, will take effect in the *same* plane.

To trace the curve of deflection in diagram *E*, we may draw a horizontal tangent through the central point of the curve, and set off from this tangent (instead of the chord line) the several deflection ordinates, which can then be easily calculated, because they are all proportional to the moment of the intercepted area in the diagram (*C*) of bending moments. At each end of the span the deflection ordinate measured in this way from the tangent, will, of course, be the same thing as the central deflection δ measured from the chord.

Each half of diagram (*C*) will have the area $M_B k = n_B k^3$; and the centre of gravity of that area will be situated at a point whose horizontal distance from the abutment is again exactly k; so that the deflection ordinate at each end of the span, measured from the tangent, or the central deflection δ measured from the chord, will have the value—

$$\delta = M_B \frac{k^2}{EI} = n_B \frac{k^4}{EI} \quad . \quad . \qquad . \quad . \quad . \qquad (3)$$

in which E is the modulus of elasticity and I the moment of inertia of the beam.

If the curve of deflection is traced out in detail, it will be found that the deflection y, measured from the chord at any other point in the span, is expressed by—

$$y = \delta \,.\, \sin \frac{x}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

so that the curve of diagram *E*, which is the actual curve of the bent pipe, will be a curve of sines, like diagrams *A* and *C*.

In this curve of deflection the radius R is not uniform, the curve being sharpest at the centre and almost straight at the extreme ends; but differentiating twice the function $y = \delta \, . \sin \frac{x}{k}$, we may find the so-called "curvature," or—

$$-\frac{1}{R} = \frac{\delta}{k^2} \, . \sin \frac{x}{k} = \frac{M}{EI} \qquad . \quad . \quad . \quad (5)$$

so that the curvature $\frac{1}{R}$ is also proportional to $\sin x$; and we know that in the curved tube the normal displacing pressure is proportional to the curvature.

Hence it will be seen that the curve of sines is exactly the kind of curve that will satisfy all the mechanical conditions. We commonly regard the curvature of a beam as the *effect* of certain transverse loads; but in this case the curvature determines the load at each point, and must at each point be proportional to the intensity of the load.

Finally, if the tube is free from any longitudinal stress, the intensity of the normal displacing force (per lineal unit) will be $n = \frac{P}{R}$, in which P is the axial fluid stress.

Hence, from (2), the bending moment due to any axial stress P will be $M = \frac{Pk^2}{R}$; while the moment of resistance evoked by the curvature $\frac{1}{R}$ will be $M = \frac{EI}{R}$; and these will be equal only when—

$$P = \frac{EI}{k^2} = EI\frac{\pi^2}{l^2} \qquad \text{Q.E.D.}$$

END OF PART I.

INDEX

A

Abutments of charged tubular arch, 55-58
Annular bridge caissons, 99
—— ——, stability of, 101
Arches, fluid. *See* Fluid
Atmospheric pressure, effect of, 9
Axial fluid stress, 21
—— —— in closed cylinders, 23
—— —— in polygonal bends, 23-27
—— —— in circular bends, 28-30
—— —— in straight pipes, 44-53
—— —— in loaded arches, 54-62

B

Bending moments in curved pipes, 35-38
—— —— in pontoons, 107-129
Bridge caissons, stability of, 86, 99
—— pressure, 47, 50, 139
Buckling tendency in straight pipes, 44, 136
—— —— in fluid arches, 140
Buoyancy in floating bodies, 81
——, curve of, in cylindrical pontoon, 90
——, ——, in rectangular pontoon, 94-96
——, ——, in cylinder on end, 98
——, ——, in annular caisson, 101

C

Caissons, annular, 99
——, circular, 98
——, with water-ballast, 104-106
Catenary curve for fluid arch, 61, 142
Centre of action for uniform fluid pressure, 10-13
Centre of action for hydrostatic pressure, 71
—— —— in regular figures just submerged, 72
—— —— in surfaces of irregular figure, 74
—— —— in surfaces submerged in deep water, 75
—— —— upon dock-gates, 77
—— —— upon inclined surfaces, 79
—— of buoyancy, 85
—— —— in submerged vessels, 86
—— —— in floating gas-holders, 87
—— —— in cylindrical pontoons, 89
—— —— in rectangular pontoons, 93
—— —— in flexible pontoons, 111
Charged tubular arches, 55, 59, 61, 142
Cigar ship, 90
Circular bends, 28, 35
—— caissons, 86, 98
Closed cylinders, 23
Components of fluid pressure, 14
Contrary fluid pressures, 76
Curve of buoyancy in cylindrical pontoon, 89
—— —— in rectangular pontoon, 93-96
—— —— in circular caissons, 99, 101
—— of deflection. *See* Deflection

D

Deflection of floating pontoons, 107
—— with central load, 110
—— with a pair of equal loads, 115-119
—— with a distributed load, 121
—— of pontoon and of ship, 122-128
Determination of the buckling pressure in pipes, 48
Diagram of hydrostatic pressures, 67
—— of buoyant pressures, 85

Diagram of moments for curved pipe, 36
—— —— for loaded pontoons, 110, 115, 121, 124, 127, 131
—— of righting moments, 97
Direction of fluid pressure, 5
Displacing forces, 23
—— ——, in polygonal bends, 23–27
—— ——, in segmental bends, 29, 30
—— ——, neutralized by longitudinal tension, 32
—— ——, generated by elastic deflection, 45
—— ——, experimentally measured, 135–140
Distribution of stress in cylinders, 20, 21
—— of buoyant pressure in floating vessels, 84
Diving machines, 83, 102
Dock caissons, 72
—— gates, 76

E

Effective axial stress, 34, 51, 63
Elastic deflection. *See* Deflection
Elastic reaction of pipe under buckling, 46
—— pontoons. *See* Deflection
Equilibrated fluid arches, 59, 142
Equilibrium of straight pipe under pressure, 46–51
—— of curved pipe, 62
—— of vertical forces in floating bodies (*see also* Stability), 81, 82, 85
Expansion joints, 33, 37, 64
Experimental measurement of displacing forces, 135
—— —— of buckling pressure, 136–140
Experiments on tensile strength of socket joints, 145

F

Flange bolts, stresses in, 42
Flange-jointed pipes, 63
Floating vessels, 81
—— ——, stability of, 81–106
Fluid arches, 54–62, 135
—— bowstrings, 60
—— catenary, 61, 142
Fluid columns, 44, 53
—— voussoirs, 25

G

Gas-holder, stability of, 86
Graphic summation of components of pressure, 14–18
—— measurement of displacing forces 24–30
—— illustration of buoyant forces, 84
Guard bolts, 39

H

Hoop-tension in cylinders, 19
Horizontal thrust in tubular arches, 56–60
Hydrostatic pressure, 7, 67
—— ——, measurement of, 68–80
—— ——, on walls of irregular figure, 72

I

Internal stress in fluids, 5

J

Joints, direct tensile strength of, 145–149
—— of polygonal bend, 27
——, riveted, 43
——, transverse, strength of, 41
——, turned and bored, 66

L

Lines of pipe, stability of, 63
Longitudinal stress in pipes, 32, 51, 62, 63
—— —— in tubular arches, 55–58

M

Mean hydrostatic pressure, 68
—— tensile stress in cylindrical shells, 20

INDEX.

Measurement of total hydrostatic pressure, 67–70
Metacentre in annular caissons, 101
—— in circular caissons, 99
—— in cylindrical pontoons, 89
—— in rectangular pontoons, 94
—— in submerged tanks and in diving machines, 103
—— in pontoons with water ballast, 105

N

Negative metacentric height, 101

P

Parabolic tubular arch, 57–60
Pipes fitted with expansion joints, 34, 37, 64
Polygonal bends, 23–27
Pontoons, stability of, 89–104
——, bending stresses in, 107–133
——, carrying a short ship, 129
Pressure on dock caissons, 72
—— on lock-gates, 76

R

Righting moment, 86, 97
Riveted joints, transverse strength of, 43
—— ——, tensile strength of, 63

S

Segmental bends, 29, 30, 35
—— tubular arches, 58–60
Socket joints, tensile strength of, 64–66, 145–149
Stability of lines of pipe, 63
—— of gas-holders, 86
—— of caissons, 86, 99–101
—— of submerged vessels, 86, 102
—— of pontoons, 89–96
Stress in cubical vessels, 21
—— in closed cylinders, 23
—— in steam-pipes of marine engines, 37
—— in bolts of flange-joints, 41
—— in tubular arches, 58
Strength of thick cylinders, 20, 21

T

Tangential stress, 19
Tanks, submerged, 102
Tensile strength of joints, 64–66, 145–149
Transverse strength of cylinders, 40
—— —— of joints, 41
Tubular arches, 54–62
Turned and bored joints, 66

U

Uniform pressure, 6

V

Vertical movement in floating bodies, 84

W

Walls of irregular figure, 72
Water-ballast in compartments, 104

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